4H-5E FURTHER COMPLEX ANALYSIS 2025

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Lecture 2.1 (revision)

1. What is Complex Analysis?

The central objects of this course are functions $f : \mathbb{C} \to \mathbb{C}$ which are holomorphic at $z \in \mathbb{C}$. This means that the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}, \qquad h \in \mathbb{C}$$

exists. This turns out to be a much stronger condition that asking that f be differentiable when viewed as real function on $\mathbb{R}^2$ by writing any complex number as $x + iy$ with $x, y \in \mathbb{R}$, and the resulting holomorphic functions have many beautiful and surprising features. For example, later in the course we will prove:

Theorem. *A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ which is zero on an open subset of $\mathbb{C}$ is zero on the whole of $\mathbb{C}$.*

Clearly this would be false for real differentiable functions and indicates that holomorphic functions are far more rigid.

2. Complex Numbers

2.1. **Cartesian coordinates.** The set of complex numbers is

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$$

where i satisfies the relation $i^2 = -1$.

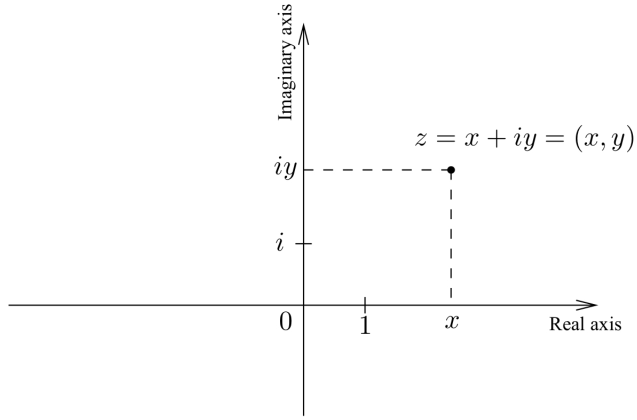


Figure 1. The complex plane

As an $\mathbb{R}$ vector space the complex numbers is isomorphic to $\mathbb{R}^2$. However, they have more structure. For example, $\mathbb{C}$ is also a ring: if $z = x + iy$ and $w = a + ib'$ then

$$zz' = (x + iy)(x' + iy') = (xx' - yy') + i(xy' + x'y)$$

If we express w as a row vector (a, b) then the above multiplication by z corresponds to the linear map

$$(a, b) \mapsto (a, b)\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = (xa - yb, ay + bx) \tag{2.1}$$

If $z = x + iy$ then we say $\mathrm{Re}(z) = x$ is the real part of z and $\mathrm{Im}(z) = y$ is the imaginary part of z. If $z = x + iy$ then the complex conjugate $\overline{z}$ of z is defined by

$$\overline{z} = x - iy$$

2.2. **Polar coordinates.** Using basic trigonometry we can also write any complex number $z = x + iy \in \mathbb{C}$ as

$$z = |z|(\cos\theta + i\sin\theta)$$

where $|z| = \sqrt{x^2 + y^2}$ and θ is the argument $\arg(z)$:

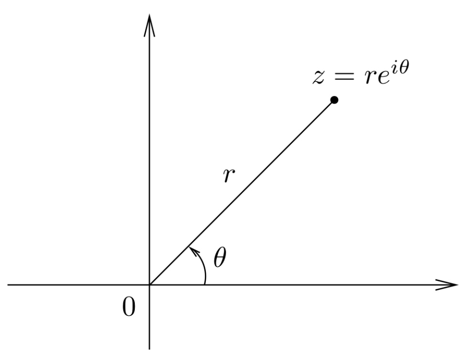


Figure 2. The polar form of a complex number

Using Eulers theorem gives: for every $\theta \in \mathbb{R}$ one has

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Therefore, every complex number can be written as

$$z = |z|e^{i\arg(z)}$$

This shows that, in terms of polar coordinates, multiplication is given by

$$zz' = |z||z'|e^{i(\arg(z) + \arg(z'))}$$

In other words, multiplication by z identifies with rotation by $\arg(z)$ composed with a dilation by $|z|$.

2.3. **Metric structure.** Using Cartesian coordinates to identify $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\} \cong \mathbb{R}^2$ we can transfer the usual metric on $\mathbb{R}^2$ to $\mathbb{C}$. In other words, the distance between two complex numbers z, w is

$$d(z, w) = |z - w|$$

This gives the notion of open and closed disks of radius $r > 0$ centred around $z \in \mathbb{C}$:

$$D_r(z) = \{w \in \mathbb{C} \mid |w - z| < r\}$$

and

$$\overline{D}_r(z) = \{w \in \mathbb{C} \mid |w - z| \leq r\}$$

The $D_r(z)$ form a basis of the topology on $\mathbb{C}$ induced by this metric. Therefore

- A subset $U \subset \mathbb{C}$ is open if for any $z \in U$ there exists an $r > 0$ such that $D_r(z) \subset U$.
- A subset $V \subset \mathbb{C}$ is closed if $\mathbb{C} \setminus V$ is open.

In particular, a subset $V \subset \mathbb{C}$ is closed if the limit of every convergent sequence in V is contained in V.

2.4. **Convergence properties.** Recall the notion of convergent and Cauchy sequences

Definition 2.2.
- A sequence $z_n \in \mathbb{C}$ converges to $z \in \mathbb{C}$ if $|z_n - z| \to 0$ as $n \to \infty$.
- A sequence $z_n \in \mathbb{C}$ is Cauchy if for every $\epsilon > 0$ there exists an $N > 0$ such that $|z_n - z_m| < \epsilon$ for all $n, m > N$.

Theorem 2.3. *$\mathbb{C}$ is a complete metric space (i.e. every Cauchy sequence is convergent)*

Proof. Suppose $z_n \in \mathbb{C}$ is a Cauchy sequence. Then $\mathrm{Re}(z_n)$ and $\mathrm{Im}(z_n)$ are Cauchy sequences in $\mathbb{R}$ because the triangle inequality implies

$$|\mathrm{Re}(z_n) - \mathrm{Re}(z_m)| \leq |z_n - z_m|$$

and likewise for the imaginary part. Since $\mathbb{R}$ is a complete metric space both sequences $\mathrm{Re}(z_n)$ and $\mathrm{Im}(z_n)$ converge to limits x and hy in $\mathbb{R}$. If $z = x + iy$ then

$$|z_n - z| \leq |\mathrm{Re}(z_n) - x| + |\mathrm{Im}(z_n) - y|$$

(again by the triangle inequality) converges to zero $n \to 0$. ☐

Definition 2.4. If $z_n \in \mathbb{C}$ is a sequence then the series $\sum_{n=0}^{\infty} z_n$ converges if the sequence of partial sums

$$S_m = \sum_{n=0}^{m} z_n$$

is convergent in $\mathbb{C}$. The series $\sum_{n=0}^{\infty} z_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |z_n|$ is convergent in $\mathbb{R}$.

Corollary 2.5. *Any absolutely convergent series is convergent.*

Proof. Any absolutely convergent series is Cauchy and therefore converges by Theorem 2.3. ☐

Example 2.6.
- For $z \in \mathbb{C}$ define
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
For $z \in \mathbb{R}$ this definition coincides with the usual exponential function. Since $|\frac{z^n}{n!}| = \frac{|z|^n}{n!}$ the absolute converges of the exponential on $\mathbb{R}$ implies absolute convergence on $\mathbb{C}$ and hence e^z converges on $\mathbb{C}$.
- For $z \in \mathbb{C}$ define
$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$
The same argument as for e^z shows there series converge absolutely and so converge. From this one deduces Euler's formulas:
$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2}$$

Lecture 2.2

3. Complex differentiability

Here we define the key concept in this course:

Definition 3.1. Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function. Then f is holomorphic (alternatively, complex differentiable) at $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \qquad z \in U$$

exists. In this case the limit is denoted $f'(z)$ and called the derivative of f at z_0.

Definition 3.2. Let $U \subset \mathbb{C}$ be an open set. Then a function $f : U \to \mathbb{C}$ is holomorphic on U if it is holomorphic at every point $z_0 \in U$.

Example 3.3.
- The constant function is holomorphic on $\mathbb{C}$.
- The identity function $f(z) = z$ is holomorphic on $\mathbb{C}$ with derivative the constant function 1.
- Complex conjugation $f(z) = \overline{z}$ is not complex differentiable at any point $z_0 \in \mathbb{C}$. Indeed

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\overline{h}}{h}, \qquad h = z - z_0$$

  However, the limit of $\frac{\overline{h}}{h}$ does not exist at $h \to 0$ because if h converges to zero along the real axis then $\frac{\overline{h}}{h} = 1$ but if h converges to zero along the imaginary axis then $\frac{\overline{h}}{h} = -1$.

The next lemma confirms that complex differentiation satisfies all the usual basic properties that are familiar from differentiation of real functions in one variable.

Lemma 3.4. *Let f and g be holomorphic functions on an open subset $U \subset \mathbb{C}$. Then complex differentiation satisfies the following:*

(1) *Linearity: $f + g$ is holomorphic on U and $(f + g)' = f' + g'$.*
(2) *Product rule: fg is holomorphic on U and $(fg)' = fg' + f'g$.*
(3) *Quotient rule: $f(z_0) \neq 0$ for $z_0 \in U$ then $1/f$ is holomorphic at z_0 and $(\frac{1}{f})'(z_0) = \frac{g'(z_0)}{g(z_0)^2}$.*
(4) *Chain rule: $f : U_1 \to U_2$ and $g : U_2 \to U_3$ then $g \circ f$ is holomorphic on U_1 and*

$$(g \circ f)'(z) = \big(g' \circ f(z)\big) f'(z)$$

for $z \in U_1$.

Proof. One argues exactly as in the case of functions of one real variable. ☐

Corollary 3.5. *If $p, q \in \mathbb{C}[X]$ are complex polynomials then $f(z) = \frac{p(z)}{q(z)}$ is complex differentiable at any $z_0 \in \mathbb{C}$ with $q(z_0) \neq 0$*

4. Recalling real differentiability

Here we begin to explore the difference between complex differentiation of $f : \mathbb{C} \to \mathbb{C}$ and real differentiability of the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = (\operatorname{Re} f(x + iy), \operatorname{Im} f(x + iy))$$

Definition 4.1. Let $F = (F_1, \dots, F_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then F is real differentiable at $x \in \mathbb{R}^n$ if there exists a linear map

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\frac{||F(x + h) - F(x) - L(h)||}{||h||} \to 0‘$$

as $h \to 0$ in $\mathbb{R}^n$. If such L exists then all the partial derivatives $\frac{\partial F_i}{\partial x_j}$ exist and L is given by the Jacobian of F

$$L(x_1, \dots, x_n) = (x_1, \dots, x_n) \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial F_1}{\partial x_n} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}$$

One obvious difference between complex differentiability is that the complex derivative is a complex number while the real derivative is a matrix. However, we have seen in (2.1) that only specific linear transformations of $\mathbb{R}^2$ correspond to multiplication by a complex number, namely those of the form $\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$. We will see shortly that complex differentiable functions are those whose real derivative identifies with multiplication by a complex number.

Example 4.2. We have already seen the function $f(z) = \overline{z}$ is not complex differentiable anywhere. However, as a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ it is given by

$$(x, y) \mapsto (x, -y)$$

which, being linear, is real differentiable with derivative $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Since $1 \neq -1$ this linear transformation does not identify with multiplication by any complex number.

5. The Cauchy–Riemann equations

Here we make the observations from the previous section precise.

Proposition 5.1. *Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ holomorphic at $z \in U$. Then f is real differentiable at z and its partial derivatives satisfy the* Cauchy–Riemann equations

$$\frac{\partial u}{\partial x}(z) = \frac{\partial v}{\partial y}(z), \qquad \frac{\partial u}{\partial y}(x) = -\frac{\partial v}{\partial x}(z) \tag{5.2}$$

where $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. Moreover

$$f'(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z)$$

In other words, f is holomorphic if it is real differentiable and its real derivative $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$ identifies with multiplication by a complex number.

Proof. For $x, y \in \mathbb{R}$ write $F(x, y) = (u(x, y), v(x, y))$ and write $f'(z) = a + ib$. As f is holomorphic we know

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - f'(z)h}{h} \to 0$$

This implies

$$\lim_{h\to 0} \frac{||F(z+h) - F(z) - f'(z)h||}{||h||} \to 0$$

and so to show F is real differentiable we only need to observe that $L(h) := f'(z)h$ defines a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. But we saw that that was the case in (2.1) and that the matrix of this linear map is $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. We conclude that $L(h)$ is the derivative of F and so, since this derivative is given by the Jacobian of F, we have

$$\begin{pmatrix} \frac{\partial u}{\partial x}(x_0, y_0) & \frac{\partial v}{\partial x}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) & \frac{\partial v}{\partial y}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

□

Lecture 3.1

6. Cauchy Riemann equations

Recall that if $f : \mathbb{C} \to \mathbb{C}$ has real part u and imaginary part v then the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are, if they exist, defined as the limits

$$\frac{\partial u}{\partial x}(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(z+h)-u(z)}{h}$$
$$\frac{\partial v}{\partial y}(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(z+ih)-u(z)}{h}$$

and likewise with v replaced by u.

Theorem 6.1. *Let $U \subset \mathbb{C}$ be an open subset and $f : U \to \mathbb{C}$ a function. Set $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. Suppose that*

(1) the first partial derivatives of u and v exist on U,
(2) are continuous on U,
(3) and satisfy the Cauchy–Riemann equations
$$\frac{\partial u}{\partial x}(z) = \frac{\partial v}{\partial y}(z), \qquad \frac{\partial u}{\partial y}(z) = -\frac{\partial v}{\partial x}(z)$$
at $z \in U$.

Then f is complex differentiable at $z \in U$ and

$$f'(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z)$$

Remark 6.2. Later we will prove a converse to Theorem 6.1. More precisely, we will show that if f is holomorphic in an open neighbourhood of $z \in U$ then conditions (1)-(3) hold. We have already seen f holomorphic implies (2) and (3) in Proposition 5.1. What is more difficult is to show that if f is holomorphic in a neighbourhood of U then the partial derivatives of u and v are continuous in a neighbourhood of U.

The following example shows why it is not enough that f satisfy the Cauchy–Riemann equations at $z \in U$ in order to deduce complex differentiability (the continuity of the partial derivatives of u and v is also important).

Example 6.3. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Then f satisfies the Cauchy–Riemann equations at $z = 0$ because

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(h)}{h} = \frac{h^5}{h^5} = 1$$

while

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(ih)}{h} = \frac{i^5(h)^4}{|h|^4} = i$$

If $f(z) = u(z) + iv(z)$ then $\frac{\partial u}{\partial x}$ is the real part of the first limit and $\frac{\partial v}{\partial y}$ is the imaginary part of the second. Thus

$$\frac{\partial u}{\partial x}(0) = 1 = \frac{\partial v}{\partial y}(0)$$

and similarly $\frac{\partial v}{\partial x}(0) = 0 = -\frac{\partial u}{\partial y}(0)$. However, f is not complex differentiable at $z = 0$. If it was complex differentiable then Proposition 5.1 would imply

$$f'(0) = 1$$

However, we can write $\frac{f(z)}{z} = \frac{z^2}{\bar{z}^2}$ and so

$$\frac{f(h(i+1))}{h(i+1)} = \frac{(1+i)^2}{(1-i)^2}$$

for $h \in \mathbb{R}$. Therefore $\lim_{\substack{h(i+1)\to 0 \\ h\in\mathbb{R}}} \frac{f(h(i+1))-f(0)}{h(i+1)} = \frac{(i+1)^2}{(1-i)^2} \neq 1$. The reason that Theorem 6.1 does not apply here is because the partial derivatives of u and v are not continuous at 0. To see this use the binomial expansion to write $(x+iy)^5 = x^5 + 5ix^4y - 10x^3y^2 - 10ix^2y^3 + 5xy^4 + iy^5$. Then

$$u(x+iy) = \begin{cases} \frac{x^5-10x^3y^2+5xy^4}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and (using the quotient rule) we compute

$$\frac{\partial u}{\partial x}(x,y) = \begin{cases} \frac{5x^4-30x^2y^2+5y^4}{(x^2+y^2)^2} - \frac{(x^5-10x^3y^2+5xy^4)4x(x^2+y^2)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

However, this function is not continuous at $(0,0)$ because $\frac{\partial u}{\partial x}(0,y) = 5$ and so $5 = \lim_{y\to 0} \frac{\partial u}{\partial x}(0,y) \neq \frac{\partial u}{\partial x}(0,0)$.

The assumption that the partial derivatives of f are continuous are required to apply the lemma:

Lemma 6.4. *Let $U \subset \mathbb{R}^n$ be open and $\phi : U \to \mathbb{R}^m$ a function with continuous partial derivatives. Then ϕ is real differentiable.*

Proof. First assume $m = 1$ and, for any vector $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ write $\alpha_i(h) = (h_1, \ldots, h_i, 0, \ldots, 0)$. Also write $D_i\phi = \frac{\partial \phi}{\partial x_i}$.

Let $a \in U$ and set $L = (D_1\phi(a), \ldots, D_n\phi(a))$. Continuity of $D_i\phi$ around a ensures that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|D_i\phi(x_i) - D_i\phi(a)| \leq \epsilon$$

whenever $||x_i - a|| \leq \delta$. The triangle inequality gives

$$\begin{aligned} |\phi(a+h) - \phi(a) - hL^t| &= |\sum_{i=1}^n (\phi(a+\alpha_i(h)) - \phi(a+\alpha_{i-1}(h)) - D_i\phi(a)h_i)| \\ &\leq \sum_{i=1}^n |\phi(a+\alpha_i(h)) - \phi(a+\alpha_{i-1}(h)) - D_i\phi(a)h_i| \end{aligned}$$

where we write L^t for the column vector obtained by transposing the row vector L. The mean value theorem asserts the existence of x_i on the line segment between $a + \alpha_i(h)$ and $a + \alpha_{i-1}(h)$ so that

$$\phi(a+\alpha_i(h)) - \phi(a+\alpha_{i-1}(h)) = D_i\phi(x_i)h_i$$

If h is sufficiently small then we must have $||x_i - a|| \leq \delta$ and so

$$\begin{aligned} |\phi(a+h) - \phi(a) - hL^t| &\leq \sum_{i=1}^n |D_i\phi(x_i)h_i - D_i\phi(a)h_i| \\ &= \sum_{i=1}^n |D_i\phi(x_i) - D_i\phi(a)||h_k| \\ &\leq \epsilon n \max |h_i| \end{aligned}$$

This proves that ϕ is real differentiable with derivative L^t. The case $m > 1$ follows easily from the case $m = 1$. ☐

Proof that conditions (1)-(3) in Theorem 6.1 imply holomorphy. Set $F(x,y) = (u(x+iy), v(x+iy))$ and suppose $z \in U$. Lemma 6.4 implies that F is real differentiable at z and so

$$\lim_{h\to 0} \frac{||F(z+h) - F(z) - L(h)||}{||h||} \to 0$$

for $L(x,y) = (x,y)\begin{pmatrix} \frac{\partial u}{\partial x}(z) & \frac{\partial v}{\partial x}(z) \\ \frac{\partial u}{\partial y}(z) & \frac{\partial v}{\partial y}(z) \end{pmatrix}$. Since u and v satisfy the Cauchy–Riemann equations it follows from (2.1) that

$$(\frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial y}(z))(w_0 + iw_1) = L(w_0, w_1)$$

Therefore

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - (\frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial y}(z))h}{h} \to 0$$

In other words, f is complex differentiable at z with derivative $f'(z) = (\frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial y}(z))$. □

Lecture 3.2

7. Power series

Definition 7.1. A power series over $\mathbb{C}$ is a series of the form $\sum_{n=0}^{\infty} a_n z^n$ with $a_n \in \mathbb{C}$.

Such power series are a large source of holomophic functions, at least on regions where they converge.

Example 7.2. The series $\sum_{n=0}^{\infty} z^n$ has sequence of partial sums converging to $\frac{1}{1-z}$ whenever $z \in \mathbb{C}$ with $|z| < 1$. Indeed $1 + z + \ldots + z^{n-1} = \frac{1-z^n}{1-z}$. Since $z^n \to 0$ as $n \to \infty$ whenever $|z| < 1$ it follows that $1 + z + \ldots + z^{n-1} \to \frac{1}{1-z}$ as $n \to \infty$ whenever $|z| < 1$.

Theorem 7.3. *For any power series $\sum_{n=0}^{\infty} a_n z^n$ there exists $0 \leq R \leq \infty$ such that*

(1) If $|z| < R$ then the series converges absolutely.
(2) If $|z| > R$ the series diverges.

Morevoer, if we use the convention that $1/0 = \infty$ and $1/\infty = 0$ then

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

Proof. We refer to Theorem 2.5 and its proof from Lectures in analysis, Volume II, Complex Analysis by Stein and Shakarchi. □

Remark 7.4. Notice that this theorem does not give information about the convergence or divergence of the series on the boundary $\overline{D}_R(0) - D_R(0) = \{z \in \mathbb{C} \mid |z| = R\}$. In fact, it is possible that the series is divergent on the whole boundary, is convergent on the whole boundary, or is only convergent on part of the boundary.

Theorem 7.5. *Any power series $\sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $D_R(0)$ where R denotes the radius of convergence. The power series*

$$\sum_{n=0}^{\infty} n a_n z^{n-1}$$

has radius of convergence R and $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ for any $z \in D_R(0)$.

Proof. Since $\lim_{n\to\infty} n^{1/n} = 1$ we have

$$\limsup |a_n|^{1/n} = \limsup |na_n|^{1/n} = \limsup |na_n|^{1/(n+1)}$$

Therefore Theorem 7.3 shows that $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ and $f(z)$ have the same radius of convergence. To prove that $g(z)$ is the complex derivative of $f(z)$ at $z \in D_R(0)$ choose $|z| < r < R$. We have to show

$$\frac{f(z+h) - f(z)}{h} - g(z) \to 0$$

as $h \to 0$. For this we are going to use the bound:

Claim. *Let $n, \delta > 0$ and suppose $h, z \in \mathbb{C}$ with $|h| < \delta$. Then*

$$|(z+h)^n - z^n - hz^{n-1}| \leq \left|\frac{h}{\delta}\right|^2 (|z| + \delta)^n$$

Proof of Claim. Using the binomial formula we have

$$(z+h)^n - z^n - hnz^{n-1} = h^2 \sum_{k=2}^{n} z^{n-k} h^{k-2} \binom{n}{k}$$

Therefore, the triangle inequality gives

$$\begin{aligned} |(z+h)^n - z^n - hz^{n-1}| &\leq \sum_{k=2}^{n} |z|^{n-k} \delta^k \binom{n}{k} \left|\frac{h}{\delta}\right|^2 \\ &\leq \left(\sum_{k=0}^{n} |z|^{n-k} \delta^k \binom{n}{k}\right) \left|\frac{h}{\delta}\right|^2 \\ &= (|z| + \delta)^n \left|\frac{h}{\delta}\right|^2 \end{aligned}$$

□

Now return to the proof of the theorem. The claim gives, for any $\delta > 0$ with $|h| < \delta$,

$$\begin{aligned}\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| &\leq \left|\frac{1}{h}\sum_{n=1}^{\infty}\left(a_n((z+h)^n - z^n - hnz^{n-1})\right)\right| \\ &\leq \frac{1}{|h|}\sum_{n=1}^{\infty}|a_n|\left|\frac{h}{\delta}\right|^2(|z|+\delta)^n \\ &= \frac{|h|}{\delta^2}\sum_{n=1}^{\infty}|a_n|(|z|+\delta)^n\end{aligned}$$

Now choose δ_1 so that $|z|+\delta_1 < R$ for R the radius of convergence of f. Then $\sum_{n=1}^{\infty}|a_n|(|z|+\delta_1)^n$ converges to a real number $C > 0$. Therefore, for any $\epsilon > 0$ we have

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \leq |h|\frac{C}{\delta_1^2} \leq \epsilon$$

whenever $|h| < \min\left(\delta_1, \epsilon\frac{\delta_1^2}{C}\right)$. In other words, $\frac{f(z+h)-f(z)}{h} - g(z) \to 0$ as $h \to 0$ as desired.

□

Corollary 7.6. *A power series is infinitely complex differentiable on its disk of convergence and the derivatives are computed by termwise differentiation.*

8. Analytic functions

More generally one can consider power series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $z_0 \in \mathbb{C}$. Such a series is said to be centred around $z_0 \in \mathbb{C}$ and Theorem 7.3 and Theorem 7.5 also applies to these more general series. This can be deduced easily by observing that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ can be written as $f_1 \circ g$ with $g(z) = z - z_0$ and $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$. In particular, the chain rule shows that

$$f'(z) = f_1'(z - z_0)$$

Definition 8.1. Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a function. Then f is analytic at $z_0 \in U$ if there exists a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ with positive radius of convergence such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

in an open neighbourhood of z_0 in U.

Lecture 4.1

9. Integration along curves

Definition 9.1.
- A parametrised curve is a map $z : [a,b] \to \mathbb{C}$ where $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ is the closed interval. We say that z is smooth if it is differentiable and has continuous derivative $z'(t)$. Note that for $t = a$ (respectively, $t = b$) the complex numbers $z'(t)$ are interpreted as the left-handed (respectively right handed) limits
$$\lim_{h\to 0, h>0} \frac{z(x+h)-z(x)}{h}, \qquad \left(\text{resp. } \lim_{h\to 0, h<0} \frac{z(x+h)-z(x)}{h}\right)$$
A parametrised curve is piecewise smooth if there are $a = a_0 < a_1 < \ldots < a_n = b$ so that $z(t)$ is smooth when restricted to each $[a_{i-1}, a_i]$.
- Two parametrised curves $z_1(t) : [a_1, b_1] \to \mathbb{C}$ and $z_2(t) : [a_2, b_2] \to \mathbb{C}$ are equivalent if there exists a continuously differentiable bijection $t : [a_1, b_1] \to [a_2, b_2]$ so that
$$z_1 = z_2 \circ t$$
and $t'(s) > 0$. This last condition ensures that the curve is traced in the same direction by z_1 and z_2. In particular, $z_1(a_1) = z_2(a_2)$ and $z_1(b_1) = z_2(b_2)$.
- The end points of a parametrised curve $z : [a,b] \to \mathbb{C}$ are $f(a)$ and $f(b)$ and a curve is called closed if $f(a) = f(b)$.
- A parametrised curve $z : [a,b] \to \mathbb{C}$ is called simple (or non-intersecting) if $t \neq t'$ then $z(t) = z(t')$ if and only if $t = a$ and $t' = b$ (or $t = b$ and $t' = a$).

For brevity we will usually refer to a smooth or piece-wise smooth parametrised curve simply as a curve.

Definition 9.2. If $\gamma : [a,b] \to \mathbb{C}$ is a smooth curve and f is a continuous function on γ (i.e. on the image of γ) then we define
$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt := \int_a^b \operatorname{Re} f(\gamma(t))\gamma'(t)dt + i\int_a^b \operatorname{Im} f(\gamma(t))\gamma'(t)dt$$
If γ is instead a piecewise smooth curve then we define
$$\int_\gamma f(z)dz = \sum_{i=1}^n \int_{a_{i-1}}^{a_i} f(\gamma(t))\gamma'(t)dt$$
for any sequence $a = a_0 < a_1 < \ldots < a_n = b$ for which γ is smooth on each $[a_{i-1}, a_i]$. Note that $\int_\gamma f(z)dz$ is independent of the choice of $a = a_0 < a_1 < \ldots < a_n = b$.

Lemma 9.3. *If $\gamma_0 : [a_0, b_0] \to \mathbb{C}$ is a reparametrisation of γ, i.e. if $\gamma = \gamma_0 \circ t$ for a differentiable bijection $t : [a,b] \to [a_0, b_0]$, then*
$$\int_\gamma f(z)dz = \int_{\gamma_0} f(z)dz$$

Proof. Using substitution for real integrals gives
$$\begin{aligned}\int_{\gamma_0} f(z)dz &= \int_{a_0}^{b_0} \operatorname{Re} f(\gamma_0(t))\gamma_0'(t)dt + i\int_{a_0}^{b_0} \operatorname{Im} f(\gamma_0(t))\gamma_0'(t)dt \\ &= \int_a^b \operatorname{Re} f(\gamma_0(t(s)))\gamma_0'(t(s))t'(s)ds + i\int_a^b \operatorname{Im} f(\gamma_0(t(s)))\gamma_0'(t(s))t'(s)dt\end{aligned}$$
Here we use that t has positive orientation, i.e. that $t(a_0) = a$ and $t(b_0) = b$. The chain rule says $\gamma'(s) = \gamma_0'(t(s))t'(s)$ and so
$$\int_\gamma f(z)dz = \int_a^b \operatorname{Re} f(\gamma(s))\gamma'(s)ds + i\int_a^b \operatorname{Im} f(\gamma(s))\gamma'(s)ds$$
which equals $\int_\gamma f(z)dz$. □

Definition 9.4. For a piecewise smooth curve $\gamma : [a,b] \to \mathbb{C}$ set
$$\operatorname{length}(\gamma) = \int_b^a |\gamma'(t)|dt$$

Proposition 9.5. *Integration of continuous functions along any piece-wise smooth curve γ in $\mathbb{C}$ satisfies:*

(1) Linearity, i.e. $\int_\gamma(\alpha f(z)+\beta g(z))dz = \alpha\int_\gamma f(z)dz + \beta\int_\gamma g(z)dz$.

(2) Orientability, i.e. if γ^- is the reverse parametrisation $\gamma^-(z)=\gamma(a+b-z)$ then

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz$$

(3)

$$|\int_\gamma f(z)dz| \le \sup_{z\in\gamma}|f(z)|\operatorname{length}(\gamma)$$

Proof. (1) follows from linearity of the usual Riemann integral and (2) follows from the change of variables formula, noting that, unlike in Lemma 9.3, the orientation of γ and γ^- are opposite. For (3) write $\int_\gamma f(z)dz = re^{i\theta}$. Then

$$|\int_\gamma f(z)dz| = |\int_a^b f(\gamma(t))\gamma'(t)dt| = r = e^{-i\theta}\int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b e^{-i\theta}f(\gamma(t))\gamma'(t)dt$$

Since $\int_a^b e^{-i\theta}f(\gamma(t))\gamma'(t)t = \int_a^b \operatorname{Re}\left(e^{-i\theta}f(\gamma(t))\gamma'(t)\right)dt + i\int_a^b \operatorname{Im}\left(e^{-i\theta}f(\gamma(t)))\gamma'(t)\right)dt$ is real, it follows that $\int_a^b \operatorname{Im}\left(e^{-i\theta}f(\gamma(t))\gamma'(t)\right)dt = 0$. Since $\operatorname{Re}\left(e^{-i\theta}f(\gamma(t))\gamma'(t)\right) \le |e^{-i\theta}f(\gamma(t))\gamma'(t)|$ known properties of the real integral imply

$$\begin{aligned}|\int_\gamma f(z)dz| &= \int_a^b \operatorname{Re}\left(e^{-i\theta}f(\gamma(t))\gamma'(t)\right)dt \le \int_a^b |e^{-i\theta}f(\gamma(t))\gamma'(t)|dt = \int_a^b |f(\gamma(t))||\gamma'(t)|dt\\ &\le \sup_{t\in[a,b]}|f(\gamma(t))|\int_b^a|\gamma'(t)|dt\\ &= \sup_{z\in\gamma}|f(z)|\operatorname{length}(\gamma)\end{aligned}$$

□

Example 9.6. Let $\gamma:[0,2\pi]\to\mathbb{C}$ be the circle $\gamma(t)=re^{it}$ of radius $r>0$ and consider the integral

$$I = \int_\gamma z^n dz$$

By definition, $I = \int_0^{2\pi}\gamma(t)^n\gamma'(t)dt$. Since the derivative of e^z is e^z we have $\gamma'(t) = ire^{it}$. Using that $e^x = \cos(x)+i\sin(x)$ we find

$$\begin{aligned}I &= \int_0^{2\pi}\left(r^ne^{nit}\right)\left(ire^{it}\right)dt = ir^{n+1}\int_0^{2\pi}e^{i(n+1)t}dt\\ &= ir^{n+1}\left(\int_0^{2\pi}\cos\left((n+1)t\right)dt + i\int_0^{2\pi}\sin\left((n+1)t\right)dt\right)\\ &= ir^{n+1}\begin{cases}\int_0^{2\pi}1dt + i\int_0^{2\pi}0dt & \text{if } n=-1\\ \left[\frac{\sin((n+1)t)}{n+1}\right]_0^{2\pi} + i\left[\frac{-\cos((n+1)t)}{n+1}\right]_0^{2\pi} & \text{if } n\neq -1\end{cases}\\ &= \begin{cases}2\pi i r^{n+1} & \text{if } n=-1\\ 0 & \text{if } n\neq -1\end{cases}\end{aligned}$$

Lecture 4.2

10. Primitives

Just as with real integration it is easy to compute the integral of a complex function along a curve if one can find a primitive of that function:

Definition 10.1. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ a function. Then $F : U \to \mathbb{C}$ is a primitive of f on U if it is holomorphic on U and satisfies

$$F'(z) = f(z), \qquad z \in U$$

Example 10.2. Any polynomial function $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ admits a primitive $F(z) = a_0 z + \frac{a_1}{2} z^2 + \ldots + \frac{a_n}{n} z^{n+1}$.

We then have the following complex version of the fundamental theorem of calculus.

Theorem 10.3. *Suppose $U \subset \mathbb{C}$ is open and $F : U \to \mathbb{C}$ is a primative of $f : U \to \mathbb{C}$. Then*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

for any piece-wise smooth curve $\gamma : [a,b] \to \mathbb{C}$.

Proof. If γ is smooth then

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_a^b \frac{d}{dt} F(\gamma(t))dt \\
&= \int_a^b \frac{d}{dt} \mathrm{Re}\left(F(\gamma(t))\right)dt + i \int_b^a \frac{d}{dt} \mathrm{Im}\left(F(\gamma(t))\right)dt \\
&= \mathrm{Re}\left(F(\gamma(b)) - F(\gamma(a))\right) + i\,\mathrm{Im}\left(F(\gamma(b)) - F(\gamma(a))\right) \\
&= F(\gamma(b)) - F(\gamma(a))
\end{aligned}$$

where the second equality follows from the chain rule $\frac{d}{dt}F(\gamma(t)) = f(\gamma(t))\gamma'(t)$ and the fourth equality follow from the fundamental theorem of calculus for real valued functions.

If instead γ is only piece-wise smooth then write $[a,b] = \bigcup_{i=0}^n [a_i, a_{i+1}]$ so that the restrictions $\gamma_i : [a_i, a_{i+1}] \to \mathbb{C}$ are smooth. Then

$$\int_\gamma f(z)dz = \sum_{i=0}^n \int_{\gamma_n} f(z)dz = \sum_{i=0}^n \left(F(\gamma(a_{i+1})) - F(\gamma(a_i))\right) = F(\gamma(b)) - F(\gamma(a))$$

where the second equality follows from the theorem for smooth γ proved above. □

Corollary 10.4. *If $f : U \to \mathbb{C}$ admits a primitive on the whole of U then*

$$\int_\gamma f(z)dz = 0$$

for any piece-wise smooth closed curve $\gamma : [a,b] \to \mathbb{C}$.

Proof. Since γ is closed one has $\gamma(a) = \gamma(b)$. Thus, if F is the primitive of f then $\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0$. □

Example 10.5. The function $f(z) = z^n$ has a primitive $F(z) = \frac{z^{n+1}}{n+1}$ on $\mathbb{C}$ when $n \neq -1$. Therefore

$$\int_\gamma z^n dz = 0$$

for any closed curve in $\mathbb{C}$. In particular, this recovers our calculation from Example 9.6.

11. Goursat's Theorem

We saw last time that if γ is a closed curve in an open set $U \subset \mathbb{C}$ on which a function f has a primitive then $\int_\gamma f(z)dz = 0$.

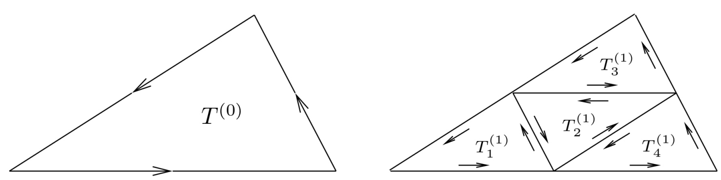
Theorem 11.1 (Goursat's Theorem)**.** *Let $U \subset \mathbb{C}$ be an open set and $T \subset U$ any Euclidean triangle whose interior is contained in U. Then*

$$\int_T f(z)dz = 0$$

for any $f : U \to \mathbb{C}$ which is holomorphic on U.

The rough idea of the proof is that, since f is holomorphic, it can be approximated by the linear function $z \mapsto f(w) + f'(w)(z-w)$ in a small neighbourhood of $w \in U$. The theorem is true for linear functions (because linear functions admit primatives) and the argument proceeds by bounding $\int_T f(z)dz$ in terms of $\int_{T'} f(z)dz$ for a triangle T' in the interior of T which is sufficiently small that in the interior of T' the approximation of $f(z)$ by $f(w) + f'(w)(z-w)$ (for some w in the interior of T') is a good approximation.

Proof. It suffices to show that $|\int_T f(z)dz| \leq \epsilon$ for any $\epsilon > 0$. We do this by subdividing the triangle. Let $T^{(0)}$ denote the original triangle and define four new triangles $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}$ and $T_4^{(1)}$ by drawing the four triangles between the vertices of $T^{(0)}$ and the the 3 midpoints of the edges of $T^{(0)}$.



With this construction the perimeter $p_j^{(1)}$ of each $T_j^{(1)}$ is $2^{-1}p^{(0)}$ and similarly the diameters (i.e. $\sup_{x,y}|x-y|$ for all x, y on T and its interior) satisfy $d_j^{(1)} = 2^{-1}d^{(0)}$. We can choose an orientation of each $T_j^{(i)}$ so that

$$\int_{T^{(0)}} f(z)dz = \int_{T_1^{(1)}} f(z)dz + \int_{T_2^{(1)}} f(z)dz + \int_{T_3^{(1)}} f(z)dz + \int_{T_4^{(1)}} f(z)dz$$

We can therefore find $j \in \{1,2,3,4\}$ with

$$|\int_{T^{(0)}} f(z)dz| \leq 4|\int_{T_j^{(1)}} f(z)dz|$$

Set $T_j^{(1)} = T^{(1)}$. Repeating this process with T^0 replaced by $T^{(1)}$ we obtain $T^{(2)} = T_j^{(2)}$ with

$$|\int_{T^{(0)}} f(z)dz| \leq 4|\int_{T_j^{(1)}} f(z)dz| \leq 4^2|\int_{T^{(2)}} f(z)dz|$$

Doing this repeatedly we obtain, by induction, triangles $T^{(n)}$ for $n \geq 0$ such that

$$|\int_{T^{(0)}} f(z)dz| \leq 4^n|\int_{T^{(n)}} f(z)dz|$$

Moreover, if $p^{(n)}$ and $d^{(n)}$ denotes respectively the perimeter and diameter of $T^{(n)}$ then we also know

$$p^{(n)} = 2^{-n}p^{(0)}, \qquad d^{(n)} = 2^{-n}d^{(0)}$$

Let $\mathfrak{T}^{(n)}$ be the union of $T^{(n)}$ together with its interior. Then $\mathfrak{T}^{(n)}$ is a closed and bounded subset of $\mathbb{C}$ and so is compact. This implies that

$$\bigcap_{n=0}^{\infty} \mathfrak{T}^{(n)} \neq \varnothing$$

so we can choose $w \in U$ so that $w \in \mathfrak{T}^{(n)}$ for each $n \geq 0$. Write

$$f(z) = f(w) + f'(w)(z-w) + \phi(z)(z-w)$$

for a function $\phi : U \to \mathbb{C}$ with $\phi(z) \to 0$ as $z \to w$ (by definition of the derivative of f at w). Then

$$\begin{aligned}\int_{T^{(n)}} f(z)dz &= \int_{T^{(n)}} \left(f(w) + f'(w)(z-w)\right) dz + \int_{T^{(n)}} \phi(z)(z-w)dz \\ &= \int_{T^{(n)}} \phi(z)(z-w)dz\end{aligned}$$

because the function $z \mapsto f(w) + f'(w)(z-w)$ is polynomial in z and hence has a primitive. Since z in this integral lies on $T^{(n)}$ and $w \in \mathfrak{T}^{(n)}$ it follows that

$$|z-w| \leq d^{(n)}$$

Therefore, part (3) of Proposition 9.5 implies

$$\left|\int_{T^{(n)}} f(z)dz\right| = \left|\int_{T^{(n)}} \phi(z)(z-w)dz\right| \leq C_n d^{(n)} p^{(n)}$$

where $C_n = \sup_{z \in T^{(n)}} |\phi(z)|$. Here we use that $p^{(n)}$ is the length of $T^{(n)}$. We conclude that

$$\left|\int_T f(z)dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)dz\right| \leq C_n 4^n d^{(n)} p^{(n)} = C_n 4^n (2^{-n} d^{(0)})(2^{-n} p^{(0)}) = C_n p^{(0)} d^{(0)}$$

Now, because $\phi(z) \to 0$ as $z \to w$ there exists $\delta > 0$ so that $|z-w| \leq \delta$ implies $|\phi(z)| \leq \frac{\epsilon}{p^{(0)} d^{(0)}}$. On the other hand, for sufficiently large n, we will have $|z-w| \leq \delta$ for all $z \in T^{(n)}$. Therefore, $C_n \leq \frac{\epsilon}{p^{(0)} d^{(0)}}$ for sufficiently large n and so $|\int_T f(z)dz| < \epsilon$. □

For applications it is also useful to having the following slight generalisation of Goursat's Theorem:

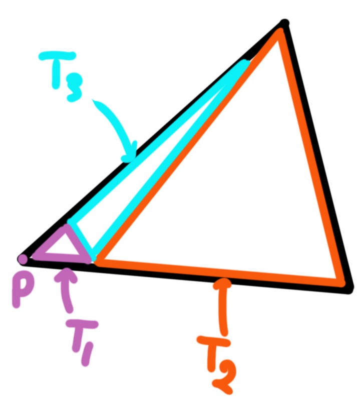
Theorem 11.2. *Let $U \subset \mathbb{C}$ be open and let $p \in U$. Suppose that $f : U \setminus \{p\} \to \mathbb{C}$ is holomorphic and bounded on an open set in U around p. Then for any triangle $T \subset U$ whose interior is also contained in U one has*

$$\int_T f(z)dz = 0$$

Proof. Let $\mathfrak{T}$ be the union of T and its interior. There are four possible cases:

Case 1. If $p \notin \mathfrak{T}$ then the theorem follows by applying Goursat's theorem to $U \setminus \{p\}$.

Case 2. If p lies on a vertex of T then we can subdivide T as follows:

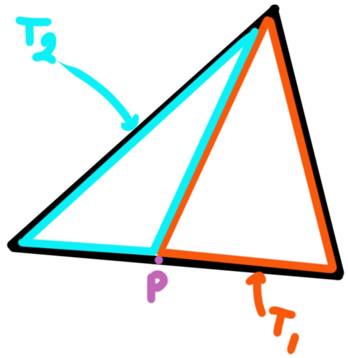


Then

$$\begin{aligned}\int_T f(z)dz &= \int_{T_1} f(z)dz + \int_{T_2} f(z)dz + \int_{T_3} f(z)dz \\ &= \int_{T_1} f(z)dz \leq \sup_{z \in T_1} |f(z)| \int_{T_1} 1dz\end{aligned}$$

where the second equality uses Goursat's theorem to show $\int_{T_2} f(z)dz = \int_{T_3} f(z)dz = 0$. Since $\int_{T_1} 1dz \to 0$ as the size of T_1 goes to zero and $\sup_{z \in T_1} |f(z)|$ is bounded by assumption, we conclude $\int_T f(z)dz = 0$.

Case 3. If p lies on a side of T then we instead subdivide T as follows:

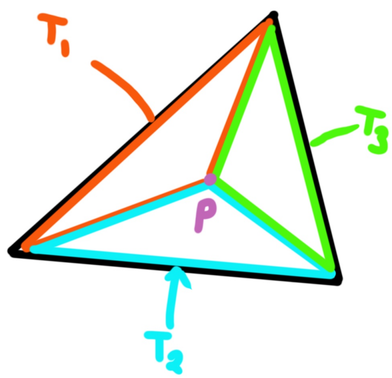


Then

$$\int_T f(z)dz = \int_{T_1} f(z)dz + \int_{T_2} f(z)dz$$

and both $\int_{T_1} f(z)dz = \int_{T_2} f(z)dz = 0$ by Case 2.

11.1. **Case 4.** If p lies in the interior of T then we can subdivide T as follows:



Then

$$\int_T f(z)dz = \int_{T_1} f(z)dz + \int_{T_2} f(z)dz + \int_{T_3} f(z)dz$$

and $\int_{T_1} f(z)dz = \int_{T_2} f(z)dz = \int_{T_3} f(z)dz = 0$ by case 2. $\square$

Lecture 5.1

12. A Local Cauchy's Theorem

Definition 12.1. A subset $U \subset \mathbb{C}$ is convex if and only if for any two points $x, y \in U$ the straight line segment between x and y is contained in U.

Example 12.2. Any open or closed ball is convex.

Theorem 12.3. *Any holomorphic function $f : U \to \mathbb{C}$ on a convex open set $U \subset \mathbb{C}$ has a primitive.*

Proof. Fix $z_0 \in U$ and, for any $z \in U$, let $\gamma(z_0, z) : [0,1] \to \mathbb{C}$ denote the straight line segment between z_0 and z. In other words,

$$\gamma(z_0, z)(t) = (1-t)z_0 + tz$$

Since U is convex we have $\gamma(z_0, z)([0,1]) \subset U$. Define

$$F(z) = \int_{\gamma(z_0,z)} f(w)dw$$

for all $z \in U$. We want to show that

$$f(z) = F'(z) := \lim_{z_1 \to 0} \frac{F(z_1) - F(z)}{z_1 - z}$$

for all $z \in U$. To do this take $z_1 \in U$ with $z_1 \neq z$. Since U is convex the triangle T between z_0, z_1 and z is contained in U. Therefore, Theorem 11.1 (Goursat's theorem) implies

$$\begin{aligned} 0 = \int_T f(w)dw &= \int_{\gamma(z_0,z_1)} f(w)dw + \int_{\gamma(z_1,z)} f(w)dw + \int_{\gamma(z,z_0)} f(w)dw \\ &= F(z_1) - F(z) + \int_{\gamma(z_1,z)} f(z)dz \end{aligned}$$

In other words, $F(z_1) - F(z) = -\int_{\gamma(z_1,z)} f(w)dw = \int_{\gamma(z,z_1)} f(w)dw$. Therefore,

$$F(z_1) - F(z) = \int_{\gamma(z,z_1)} f(w)dw = f(z)\int_{\gamma(z,z_1)} dw + \int_{\gamma(z,z_1)} (f(w) - f(z))\, dw$$

Now $\int_{\gamma(z,z_1)} dw = z_1 - z$, because of Theorem 10.3 and because the constant function has z as an anti-derivative. Therefore

$$\begin{aligned} \left| \frac{F(z_1) - F(z)}{z_1 - z} - f(z) \right| &= \left| \frac{\int_{\gamma(z,z_1)} (f(w) - f(z))dw}{z_1 - z} \right| \\ &\leq \sup_{w \in \gamma(z,z_1)} |f(w) - f(z)| \frac{\text{length}\, \gamma(z, z_1)}{|z_1 - z|} \\ &= \sup_{w \in \gamma(z,z_1)} |f(w) - f(z)| \end{aligned}$$

where the inequality uses Proposition 9.5. Since f is continuous at z it follows that $\sup_{\zeta \in \gamma(z,z_1)} |f(w) - f(z)| \to 0$ as $z_1 \to z$. This ensures

$$F'(z) := \lim_{z_1 \to 0} \frac{F(z_1) - F(z)}{z_1 - z} = f(z)$$

as required. □

Theorem 12.4 (Cauchy's Theorem for convex sets)**.** *If f is holomorphic on a convex open set $U \subset \mathbb{C}$ then $\int_\gamma f(z)dz = 0$ for every closed curve γ in U.*

Proof. Theorem 12.3 implies the existence of a holomorphic F on U with $F'(z) = f(z)$. The fundamental theorem of calculus (Theorem 10.3) therefore implies

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0$$

if γ is parametrised as $[a, b] \to \mathbb{C}$. □

13. A slight generalisation of Cauchy's theorem

The proof of Theorem 12.3 actually gives the stronger result:

Corollary 13.1. *Let $U \subset \mathbb{C}$ be a convex open subset and suppose $f : U \to \mathbb{C}$ is continuous and holomorphic on $U \setminus \{p\}$. Then f admits a primitive on $U \setminus \{p\}$.*

Proof. Since f is continuous at p it is bounded in an open neighbourhood of p. Therefore, one can repeat the proof from Theorem 12.3 but replacing the use of Goursat's theorem by the more general version in Theorem 11.2. ☐

Applying the fundamental theorem of calculus yields:

Corollary 13.2. *Let $U \subset \mathbb{C}$ be a convex open subset and suppose $f : U \to \mathbb{C}$ is continuous, and holomorphic on $U \setminus \{p\}$. Then $\int_\gamma f(z)dz = 0$ for any closed curve γ in $U \setminus \{p\}$.*

14. Winding numbers

Definition 14.1. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve and p a point not on γ. The winding number of γ around p is defined as

$$W(\gamma, p) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - p} dz = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - p} dt$$

Example 14.2. If γ is a circle $\gamma(t) = p + e^{it}$ for $t \in [0, 2\pi]$ then Example 9.6 shows that $W(\gamma, p) = 1$.

For general curves γ we should view $W(\gamma, p)$ as counting the number of times γ wraps around p. To demonstrate why this is reasonable we assume $p = 0$ for simplicity and consider the complex logarithm

$$\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z)$$

where log denotes the real logarithm. The argument Arg is only well defined up to a multiple of $2\pi i$ and this ambiguity means that Log cannot be defined as a holomorphic function on the whole of $\mathbb{C}$. Nevertheless, it can be defined locally and when defined its derivative is $1/z$ (as follows from the identity $e^{\mathrm{Log}(z)} = z$). This means the derivative of $\mathrm{Log}(\gamma(t))$ is $\frac{\gamma'(t)}{\gamma(t)}$ and so the Fundamental theorem of calculus gives:

$$\int_a^b \frac{\gamma'(t)}{\gamma(t)} dt = \mathrm{Log}(\gamma(b)) - \mathrm{Log}(\gamma(a)) = i\left(\mathrm{Arg}(\gamma(b)) - \mathrm{Arg}(\gamma(a))\right) \tag{14.3}$$

(since γ is a closed curve and the real part of $\mathrm{Log}(\gamma(t))$ has no ambiguity these reals parts cancel in the above formula). In other words, $\int_a^b \frac{\gamma'(t)}{\gamma(t)} dt$ measures the change of the argument of $\gamma(t)$ as we go around the path γ.

Warning 14.4. In (14.3) is it important to note that the formula only makes sense if we define Arg locally in a continuous way. For example, if we defined the argument to jump to back to 0 as it approaches 2π then the formula would be wrong because the right hand side would be zero).

15. Cauchy's Integral formula

Theorem 15.1. *Let f be holomorphic on an open convex set U and let γ be a piecewise smooth closed curve in U. Then*

$$W(\gamma, p) f(p) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - p} fz$$

for any $p \in U \setminus \gamma([a, b])$.

Proof. Set

$$g(z) = \begin{cases} \frac{f(z) - f(p)}{z - p} & z \neq p \\ f'(p) & z = p \end{cases}$$

Then $g(z)$ is continuous on U and holomorphic on $U \smallsetminus \{p\}$. By the improved versions of Cauchy's theorem we have $\int_\gamma g(z)dz = 0$ and so

$$\begin{aligned} 0 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(p)}{z-p}dz &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-p}dz - \frac{1}{2\pi i}\int_\gamma \frac{f(p)}{z-p}dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-p}dz - W(\gamma,p)f(p) \end{aligned}$$

□

As a special case we obtain:

Theorem 15.2 (Cauchy's integral formula)**.** *Let f be holomorphic on an open set U containing a circle γ, oriented in the counter clockwise direction, and its interior. Then*

$$f(p) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-p} fz$$

for any p in the interior of γ.

Proof. This follows from Theorem 15.1. Since γ is a circle we can find a convex open subset $U' \subset U$ which contains γ and its interior. If p lies in the interior of γ then we know $W(\gamma,p) = 1$, and so $f(p) = f(p)W(\gamma,p) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-p} fz$. □

Lecture 5.2

16. Uniform convergence and limits of integrals

Recall that if one has a sequence of functions $f_n \to f$ then in general it is a subtle question whether $\int f = \lim \int f_n$. One case where one can interchange the limit and the integral is when the $f_n \to f$ uniformly. This means:

Definition 16.1. For any subset $X \subset \mathbb{C}$ we say that a sequence of functions $f_n : X \to \mathbb{C}$ converges to $f : X \to \mathbb{C}$ uniformly if for any $\epsilon > 0$ there exists an $N > 0$ such that

$$|f_n(z) - f(z)| < \epsilon$$

for any $n > N$ and any $z \in X$. This is equivalent to asking that

$$\sup_{z\in X} |f_n(z) - f(z)| \to 0$$

as $n \to \infty$.

Lemma 16.2. *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve and suppose f_n, f are continuous functions on γ with $f_n \to f$ uniformly on $\mathrm{Im}(\gamma)$. Then*

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$$

as $z \to \infty$.

Proof. We have

$$\begin{aligned} |\int_\gamma f_n(z)dz - \int_\gamma f(z)dz| &= \int_\gamma (f_n - f)(z)dz \\ &\le \sup_{z\in\gamma} |f_n(z) - f(z)| \,\mathrm{length}(\gamma) \to 0 \end{aligned}$$

as $n \to \infty$. □

17. More on Cauchy's integral formula

Theorem 17.1 (Cauchy's integral formula for derivatives)**.** *If f is holomorphic on an open set U then f has infinitely many holomorphic derivatives. Moreover, for any circle $C \subset U$ oriented counter clockwise whose interior is also contained in U one has*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}$$

for $n \ge 0$ for all z in the interior of C.

Proof. Argue by induction. When $n = 0$ the statement is Theorem 15.2. For $n > 0$ assume that the first $n-1$-th derivatives of f exist and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{(w-z)^n} fw$$

Then

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{n-1!}{2\pi i}\int_C f(w)\left(\frac{1}{h(w-z-h)^n} - \frac{1}{h(w-z)^n}\right)dw$$

Now choose a sequence $h_N \in \mathbb{C}$ converging to zero as $N \to \infty$. To prove the theorem we have to understand

$$\lim_{N\to\infty} \frac{n-1!}{2\pi i}\int_C f(w)F_N(w)dw$$

where $F_N(w) = \frac{1}{h_N(w-z-h_N)^n} - \frac{1}{h_N(w-z)^n}$. Now

$$F_N(w) \to \frac{d}{du}\left(\frac{1}{(w-z)^n}\right) = \frac{n}{(w-z)^{n+1}}$$

as $N \to \infty$. We claim this convergence is uniform on C. This claim allows us to use Lemma 16.2 to deduce that

$$\lim_{N\to\infty} \frac{n-1!}{2\pi i}\int_C f(w)F_N(w)dw = \frac{n-1!}{2\pi i}\int_C \frac{f(w)n}{(w-z)^{n+1}}dw = \frac{n!}{2\pi i}\int_C \frac{f(w)n}{(w-z)^{n+1}}dw$$

which finishes the proof. Therefore, it only remains to check the claimed uniform convergence. This can be done by showing that:

$$\sup_{w\in C}\left|\left(\frac{1}{h_N(w-z-h_N)^n}-\frac{1}{h_N(w-z)^n}\right)-\frac{n}{(w-z)^{n+1}}\right|\to 0$$

as $N\to 0$. To see this recall the expansion

$$\frac{1}{(1-x)^n}=\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k$$

which converges for $|x|<1$. We find that

$$\begin{aligned}\frac{1}{h(w-z-h)^n}-\frac{1}{h(w-z)^n}&=\frac{1}{h(w-z)^n}\left(\frac{1}{(1-\frac{h}{w-z})^n}-1\right)\\&=\frac{n}{(w-z)^{n+1}}+h\sum_{k=2}^{\infty}\binom{n+k-1}{k}\left(\frac{h^{k-2}}{(w-z)^{n+k}}\right)\end{aligned}$$

whenever $h\in\mathbb{C}$ has $|\frac{h}{w-z}|<1$. Now C is compact and so there is a constant $A>0$ with $|\frac{1}{w-z}|<A$ for all $w\in C$. This means that

$$\left|\frac{1}{h(w-z-h)^n}-\frac{1}{h(w-z)^n}-\frac{n}{(w-z)^{n+1}}\right|\leq hA^{n+2}\sum_{k=2}^{\infty}\binom{n+k-1}{k}|hA|^{k-2}$$

whenever $|hA|<1$. In particular, if $B=A^{n+2}\sum_{k=2}^{\infty}\binom{n+k-1}{k}\frac{1}{2^{k-2}}$, then

$$\sup_{w\in C}\left|\left(\frac{1}{h_N(w-z-h_N)^n}-\frac{1}{h_N(w-z)^n}\right)-\frac{n}{(w-z)^{n+1}}\right|\leq h_NB$$

whenever $|h_NA|<1/2$. We conclude that the left hand side converges to zero as $N\to\infty$ which was what we wanted.

□

Example 17.2. Let γ be the unit circle. We can use Cauchy's integral formula for derivatives to calculate the integral

$$\int_\gamma \frac{\cos(z)}{z^3}dz$$

Applying Theorem 17.1 with $f=\cos$, $n=2$, and $z=0$ gives

$$\int_\gamma \frac{\cos(z)}{z^3}dz=f^{(2)}(0)\frac{2\pi i}{2!}=-\cos(0)\pi i=-\pi i$$

18. Cauchy's estimate

Theorem 18.1 (Cauchy's estimate)**.** *Let $U\subset\mathbb{C}$ be an open subset and $f:U\to\mathbb{C}$ holomorphic. If U contains a circle γ centered at z and of radius $R>0$, and its interior, then*

$$|f^{(n)}(z)|\leq\frac{n!\sup_{w\in\gamma}|f(w)|}{R^n}$$

Proof. Theorem 17.1 (Cauchy's derivative formula) implies

$$\begin{aligned}|f^{(n)}(z)|&=\left|\frac{(n)!}{2\pi i}\int_\gamma\frac{f(w)}{(w-z)^{n+1}}dw\right|\\&\leq\frac{n!}{2\pi i}\sup_{w\in\gamma}\left|\frac{f(w)}{(w-z)^{n+1}}\right|\mathrm{length}(\gamma)\\&=\frac{n!\sup_{w\in\gamma}|f(w)|}{R^n}\end{aligned}$$

□

19. Liouville's Theorem

Definition 19.1. A function $f : \mathbb{C} \to \mathbb{C}$ is entire if it is holomorphic on $\mathbb{C}$.

Theorem 19.2 (Liouville's Theorem)**.** *Any bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

Proof. We show that $f'(z) = 0$. This will imply $f(z)$ is constant because $\mathbb{C}$ is a connected topological space. Let $w \in \mathbb{C}$ and let γ_R be the circle around w of radius R. Then Cauchy's estimate (Theorem 18.1) implies

$$|f'(w)| \leq \frac{1}{R} \sup_{z \in \gamma_R} |f(z)|$$

Since f is bounded we have $\sup_{z \in \gamma_R} |f(z)| \leq B$ for some constant $B > 0$ and all R. Letting $R \to \infty$ shows $|f'(w)| = 0$ and so $f'(w) = 0$. $\square$

Theorem 19.3 (Fundamental theorem of algebra)**.** *Every non-constant polynomial*

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n, a_n \in \mathbb{C}$$

has a root in $\mathbb{C}$ (i.e there exists $z \in \mathbb{C}$ such that $f(z) = 0$).

Proof. Suppose that f has no roots. Then $\frac{1}{f}$ is an entire function. On the other hand, if we assume the leading coefficient a_n in $f(z)$ is non-zero then

$$f(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right)$$

and the term $\frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \to 0$ as $|z| \to \infty$. We can therefore choose an $R > 0$ such that if $|z| > R > 0$ then

$$\left| \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right| < \frac{|a_n|}{2}$$

Note that for $a, b \in \mathbb{C}$ the triangle inequality gives $|a| = |a + b - b| < |a + b| + |b|$ and so $|a| - |b| < |a + b|$. In particular, if $|b| < |a|/2$ then

$$|a|/2 < |a| - |b| < |a + b|$$

Applying this with $a = a_n$ and $b = \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}$ gives

$$\frac{|a_n||z^n|}{2} < |f(z)|$$

when $z > R$. We conclude that

$$|\frac{1}{f(z)}| < \frac{2}{|a_n||z^n|} < \frac{2}{|a_n|R^n}$$

when $|z| > R$. On the other hand, since $\frac{1}{f}$ is continuous on the compact set $\overline{D}_R(0) = \{z \in \mathbb{C} | z| \leq R\}$ it follows that $\frac{1}{f}$ is bounded on $\overline{D}_R(0)$. We conclude that $\frac{1}{f}$ is bounded on $\mathbb{C}$ and so Liouville's theorem (Theorem 19.2) implies $\frac{1}{f}$, and hence f, is constant on $\mathbb{C}$. This is a contradiction. $\square$

Lecture 6.1

20. Morera's theorem

Theorem 20.1 (Morera's Theorem)**.** *Let $U \subset \mathbb{C}$ be a convex open subset and suppose $f : U \to \mathbb{C}$ is a continuous function. If $\int_T f(z)dz = 0$ for any triangle $T \subset U$ whose interior is also contained in U then f is holomorphic.*

Proof. In the proof of Theorem 12.3 we showed that if $\int_T f(z)dz = 0$ for each triangle then, if $z_0 \in U$ is fixed, the function

$$F(z) := \int_{\gamma(z_0,z)} f(z)dz$$

(recall $\gamma(z_0, z)$ is the straight line from z_0 to z) is holomorphic on u with complex derivative $f(z)$. Cauchy's integral formula for derivatives (Theorem 17.1) then implies that $F'(z) = f(z)$ is also holomorphic. □

Corollary 20.2. *If $U \subset \mathbb{C}$ is convex and open and $f_n : U \to \mathbb{C}$ is a sequence of holomorphic functions converging uniformly to $f : U \to \mathbb{C}$ then f is also holomorphic.*

Proof. The uniform limit of continuous functions is again continuous by a problem on the exercise sheets. Therefore, f is continuous and we can deduce that f is holomorphic by showing $\int_T f(z)dz = 0$ for each triangle in T and applying Morera's theorem (Theorem 20.1). By Goursat's theorem (Theorem 11.1) we know $\int_T f_n(z)dz = 0$ and so, using Lemma 16.2, we find

$$0 = \lim_{n\to\infty} \int_T f_n(z)dz = \int_T f(z)dz$$

□

21. Taylor's theorem

Theorem 21.1 (Taylor's theorem)**.** *If $f : U \to \mathbb{C}$ is a holomorphic function on an open subset $U \subset \mathbb{C}$ and $p \in U$. Then for any open disk $D \subset U$ centred at p one has*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$$

where

$$a_n = \frac{f^{(n)}(p)}{n!}$$

Proof. Using a linear change of variable we can assume $p = 0$ (replace $f(z)$ by the function $z \mapsto f(z-p)$). Let $R > 0$ be the radius of D. Since f is holomorphic in an open neighbourhood of D Cauchy's integral formula gives

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$$

for γ the boundary of D (oriented counterclockwise). We can write

$$\frac{1}{w-z} = \frac{1}{w}\frac{1}{(1-z/w)} = \underbrace{\frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n}_{S(w)}$$

whenever $|z/w| < 1$. In particular, this is valid whenever z lies in the interior of D and $w \in \gamma$. Now fix such a z and, for $N > 0$, set

$$S_N(w) = \frac{1}{w}\sum_{n=0}^{N}\left(\frac{z}{w}\right)^n$$

Notice that for $w \in \gamma$ one has

$$
\begin{aligned}
|S(w) - S_N(w)| &= |\frac{1}{w}\sum_{n=N}^{\infty}\left(\frac{z}{w}\right)^n| \\
&= |\frac{1}{w}\left(\frac{z}{w}\right)^N \sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n| \\
&\leq \frac{|z|^N}{|w|^{N+1}}\sum_{N=0}^{\infty}|\frac{z}{w}|^n = \frac{|z|^N}{R^{N+1}}\frac{1}{(1-|z|/R)} = \frac{|z|^N}{R^N}\frac{1}{(R-|z|)}
\end{aligned}
$$

Therefore, $|S(w) - S_N(w)| \to 0$ independently of $w \in \gamma$ and so the sequence of functions S_N converges uniformly to S. It follows that $f(w)S_N(w)$ also converges uniformly to $f(w)S(w)$ and so Lemma 16.2 implies

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_\gamma f(w)S_N(w)dw = \frac{1}{2\pi i}\int_\gamma f(w)S(w)dw$$

We know already that the right hand side of this expression is $f(z)$. The left hand side is

$$\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)z^n}{w^{n+1}}dw\right) = \sum_{n=0}^{\infty} z^n\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}dw\right)$$

By Cauchy's integral formula for derivatives (Theorem 17.1) each term in the right hand sum equal $\frac{f^{(n)}(0)}{n!}$. Thus

$$f(z) = \sum_{n=0}^{\infty} z^n \frac{f^{(n)}(0)}{n!}$$

as required. □

Lecture 6.2

22. Isolated zeroes

Theorem 22.1. *Let f be a holomorphic function on a connected open set U and w_n is a non-constant sequence in U converging to $p \in U$. If $f(w_n) = 0$ then $f(z) = 0$ for all $z \in U$.*

Proof. Let D be an open disc centered at p and contained in U. By Taylor's theorem (Theorem 21.1), f admits a power series expansion on D

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$$

for any $z \in D$. Since f is continuous we have $f(p) = \lim f(w_n) = 0$. Therefore $a_0 = 0$. We want to show $a_n = 0$ also for all $n > 0$. If not we can find a smallest $m \geq 1$ with $a_m \neq 0$. Then

$$f(z) = a_m (z-p)^m \left(1 + \underbrace{\sum_{n=m+1}^{\infty} \frac{a_n}{a_m} (z-p)^{n-m}}_{g(z)} \right)$$

with $g(z)$ a convergent power series on D and so holomorphic. Since $g(p) = 0$ continuity implies the existence of a neighbourhood U_0 of p such that $|g(z)| < 1$ for all $z \in U_0$. This means $1 + g(z) \neq 0$ on U_0 and so $f(z) = a_m(z-p)^m(1+g(z)) \neq 0$ on $U_0 \setminus \{p\}$. However, this is a contradiction because by assumption $f(w_n) = 0$ and since $w_n \to p$ and is non-constant we have $w_n \in U_0$ for large n and not all w_n equal p. We conclude that $a_n = 0$ for all $n \geq 0$ and so $f = 0$ on D.

It remains to check that this implies $f = 0$ on the whole connected set U. To see this let

$$V = \{z \in U \mid f = 0 \text{ in an open neighbourhood of } z\}$$

Then V is non-empty and by definition it is also open. On the other hand V is also closed because we've just shown that if $z_n \to z$ with $f(z_n) = 0$ then f is also zero in an open neighbourhood of z and so $z \in V$. Since U is connected the only open and closed subset of U is U itself. Therefore $f = 0$ on U. ☐

23. The identity principle

Theorem 23.1 (The identity principle)**.** *If f and g are two holomorphic functions on a connected open subset $U \subset \mathbb{C}$ and the set of $z \in U$ where $f(z) = g(z)$ contains a non-constant sequence converging to a point in U then $f = g$ on U.*

Proof. Apply Theorem 22.1 to $f - g$ to deduce $f - g = 0$ on U, and so $f = g$ on U. ☐

Example 23.2. Let $f : D_2(0) \to \mathbb{C}$ be a holomorphic function such that $f(1/n) = 1/n^2$ for all $n \in \mathbb{Z}_{\geq 1}$. What is $f(i)$? To compute this set $g(z) = z^2$. Then $g(z)$ is holomorphic and the set of $z \in D_2(0)$ for which $f(z) = g(z)$ contains the convergent sequence $1/n$. Therefore, the identity principle implies $f = g$ on $D_2(0)$ and so $f(i) = i^2 = -1$.

24. Singularities

Definition 24.1. We say that f has an isolated singularity at $p \in \mathbb{C}$ if f is defined on an open neighbourhood of p but not at the point p itself. In particular, if $U \subset \mathbb{C}$ is an open subset of $\mathbb{C}$ and $p \in U$ then any function $f : U \setminus \{p\} \to \mathbb{C}$ has an isolated singularity at p.

Example 24.2. Then function $f(z) : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = 1/z$ has an isolated singularity at $z = 0$. Similarly the function $g(z) : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $g(z) = z$ has an isolated singularity.

There are three possible kinds of isolated singularity:

Definition 24.3. If f has an isolated singularity at $p \in \mathbb{C}$ then either:

(1) p is a removable singularity if f extends to a holomorphic function on a neighbourhood of p. This implies $f(z) \to f(p)$ at $z \to p$.
(2) p is a pole if $|f(z)| \to \infty$ as $z \to p$.
(3) p is an essential singularity otherwise.

Example 24.4. (1) The function $1/z$ has a pole at $z = 0$.

(2) Since $\sin z = z + z^3/3! + z^5/5! - z^7/7! + \dots$ we have

$$\sin z = z(1 + z^2/3! - \dots)$$

and so $1/\sin z = z^{-1} \underbrace{(1 + z^2/3! - \dots)^{-1}}_{(A)}$. Since A is holomorphic on an open neighbourhood of 0 it follows that $1/\sin z$ has a pole at $z = 0$.

(3) Since $e^z = 1 + z + z^2/2! + z^3/3! + \dots$ we have

$$\frac{e^z - 1}{z} = \frac{z + z^2/2! + z^3/3! + \dots}{z} = 1 + z/2! + z^2/3! + \dots$$

so $\frac{e^z-1}{z}$ has a removable singularity at $z = 0$.

(4) Consider the singularity of the function $f(z) = e^{1/z}$ at $z = 0$. This singularity is not a removable singularity because $f(1/n) = e^n$ and so $f(1/n) \to \infty$ as $n \to \infty$. However, it is also not a pole because if $w_n = 1/2\pi i n$ then $w_n \to 0$ but $|f(w_n)| = 1$. Therefore $f(z)$ has an essential singularity.

25. Riemann's theorem on removable singularities

Theorem 25.1 (Riemann's theorem on removable singularities)**.** *Let $U \subset \mathbb{C}$ be open and $p \in U$. If f is holomorphic on $U \setminus \{p\}$ and bounded then p is a removable singularity.*

Proof. Set

$$g(z) = \begin{cases} (z-p)^2 f(z) & \text{if } z \neq p \\ 0 & \text{if } z = p \end{cases}$$

Then g is holomorphic on $U \setminus \{p\}$. We also have

$$\left|\frac{g(z) - g(p)}{z - p}\right| = |z - p||f(z)|$$

Since f is bounded on $U \setminus \{p\}$ there exists an $M > 0$ such that $|f(z)| < M$ for all $z \in U \setminus \{p\}$. Therefore

$$\left|\frac{g(z) - g(p)}{z - p}\right| = |z - p||f(z)| \leq M|z - p|$$

It follows that g is holomorphic at p with $g'(p) = 0$. Let D be an open disk in U centred at p. Then Theorem 21.1 (Taylor's theorem) allows us to write

$$g(z) = \sum_{n=2}^{\infty} a_n (z-p)^n = (z-p)^2 \sum_{n=2}^{\infty} a_n (z-p)^{n-2}$$

Therefore $f(z) = \sum_{n=2}^{\infty} a_n (z-p)^{n-2}$ on $D \setminus \{p\}$. However, the series $\sum_{n=2}^{\infty} a_n (z-p)^{n-2}$ has a radius of convergence $R \geq 0$ for which it converges when $|z - p| < R$ and diverges when $|z - p| > R$ (Theorem 7.3). Since the series converges for $z \in D \setminus \{p\}$ we have R greater than or equal the radius of D. In particular, $R > 0$. Therefore Theorem 7.5 implies f is holomorphic in an neighbourhood of z and so f has a removable singularity at p. $\square$

Lecture 7.1

26. Local forms of singularities

Here we consider a holomorphic function f with an isolated singularity at $p \in \mathbb{C}$.

- If f has an *isolated zero* at p (i.e. removable singularity at p with the extension $f(p) = 0$) and is not identically zero in a neighbourhood of p then there exists a holomorphic g with $g(p) \neq 0$ and an $n \geq 1$ such that

$$f(z) = (z-p)^n g(z) \tag{26.1}$$

in a neighbourhood of p.

Proof. Since p is a removable singularity we can extend f to a holomorphic function in a disk D around p. We can then use Taylor's theorem (Theorem 21.1) to write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$$

for $z \in D$. Since $f(p) = 0$ we have $a_0 = 0$. Since f is not identically zero there is a smallest $n \geq 1$ with $a_n \neq 0$. Then

$$f(z) = (z-p)^n \sum_{m=0}^{\infty} a_{m+n}(z-p)^m$$

and we can take $g(z) = \sum_{m=0}^{\infty} a_{m+n}(z-p)^m$ which is holomorphic on D by Theorem 7.5. □

Definition 26.2. The order of an isolated zero at p is the integer n in the expression (26.1).

- If f has a pole at p then there exists a non-vanishing holomorphic function g and $n \geq 1$ such that $g(p) \neq 0$ and

$$f(z) = \frac{g(z)}{(z-p)^n} \tag{26.3}$$

on $D \setminus \{p\}$ for D a disk around p.

Proof. Since f has a pole at $z = p$ we have that $1/f(z) \to 0$ as $z \to p$. Therefore $1/|f(z)|$ is bounded in a neighbourhood of p and so Riemann's removable singularity theorem allows us to extend $1/f(z)$ to a holomorphic function $h(z)$ on a neighbourhood around p. Since $h(p) = 0$ the local form of isolated zeroes tells us that $h(p) = (z-p)^n h_1(z)$ for a holomorphic function $h_1(z)$ with $h_1(p) \neq 0$. Then $g(z) = 1/h_1(z)$ is holomorphic in a neighbourhood of p (since $h_1(p) \neq 0$) and

$$f(z) = \frac{1}{h(z)} = \frac{1/h_1(z)}{(z-p)^n} = \frac{g(z)}{(z-p)^n}$$

as claimed. □

Definition 26.4. The order of a pole at p is the integer n in the expression (26.1).

27. Residues

The local form of a holomorphic function f with a pole at p asserts that $f(z) = \frac{g(z)}{(z-p)^n}$ with g holomorphic in a neighbourhood of p and $g(p) \neq 0$. If we write $g(z) = \sum_{i=0}^{\infty} a_{-n+i}(z-p)^i$ then

$$f(z) = \frac{a_{-n}}{(z-p)^n} + \frac{a_{-n+1}}{(z-p)^{n-1}} + \ldots + \frac{a_{-1}}{(z-p)} + G(z)$$

where $G(z)$ is the holomorphic function $\sum_{i=0}^{\infty} a_i (z-p)^i$. We call $\frac{a_{-n}}{(z-p)^n} + \frac{a_{-n+1}}{(z-p)^{n-1}} + \ldots + \frac{a_{-1}}{(z-p)}$ the principal part of f at p.

Definition 27.1. The coefficient a_{-1} in the principal part of f at p is called the residue of f at p and denoted $\mathrm{res}_{z=p}(f)$.

We can also compute the residue of f directly if we write $f(z) = \frac{h(z)}{(z-p)^n}$ with $h(p) \neq 0$.

Lemma 27.2. *If* $\frac{h(z)}{(z-p)^n}$ *with* $h(p) \neq 0$ *then*

$$\mathrm{res}_{z=p}(f) = \frac{h^{(n-1)}(p)}{(n-1)!}$$

Alternatively,

$$\mathrm{res}_{z=p}(f) = \frac{1}{(n-1)!} \lim_{z \to p} \frac{d^{n-1}}{dz^{n-1}} \left((z-p)^n f(z)\right)$$

Proof. If $h(z)$ has Taylor expansion $\sum_{i=0}^{\infty} a_{-n+i}(z-p)^i$ then

$$h^{(m)}(z) = \sum_{i=0}^{\infty} a_{-n+i}(z-p)^{i-m}(i)(i-1)\dots(i-m+1)$$

Therefore

$$h^{(m)}(p) = a_{-n+m}(m)(m-1)\dots(1) = a_{-n+m}m!$$

In particular, $h^{(n-1)}(z) = a_{-1}(n-1)! = \mathrm{res}_{z=p}(f)(n-1)!$. The second expressions follows from the first since the $n-1$-th complex derivative of $(z-p)^n f(z)$ is continuous. $\square$

Example 27.3. (1) Consider $f(z) = \frac{e^z}{z^n}$. This has a pole at zero of order n. Since $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ it follows that the principal part of $f(z)$ is

$$\frac{1}{z^n} + \frac{1}{z^{n-1}} + \frac{1}{z^{n-2}2!} + \dots + \frac{1}{z(n-1)!}$$

Therefore, $\mathrm{res}_{z=0}(f) = \frac{1}{(n-1)!}$.

(2) Consider the function $f(z) = \frac{z^2+1}{z(1-z)}$. This has a pole at $z=0$ of order 1 and we can compute its residue in two says. First, one can use the binomial expansion $\frac{1}{1-z} = 1 + z + z^2 + \dots$ which converges for $|z| < 1$. Therefore,

$$\frac{z^2+1}{1-z} = (z^2+1)(1+z+z^2+\dots) = 1 + z + 2z^2 + 2z^3 + \dots$$

It follows that the principal part of $f(z)$ is $1/z$ and so $\mathrm{res}_{z=0}(f) = 1$. Alternatively, one can use the formula

$$\mathrm{res}_{z=0}(f) = \frac{1}{0!} \lim_{z \to 0} (z f(z)) = \lim_{z \to 0} \frac{z^2+1}{1-z} = 1$$

(3) Consider the function $f(z) = \frac{\sin z}{z^2(z-1)}$. Since $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$ we see that $f(z)$ has a pole of order 2 at $z=0$. Again using the binomial expansion $\frac{1}{1-z} = 1 + z + z^2 + \dots$ we see that

$$\frac{\sin z}{z^2(1-z)} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)(1+z+z^2+\dots) = \frac{1}{z^2}\left(z + z^2 + (1 - \frac{1}{3!})z^3 + \dots\right)$$

for $|z| < 1$. It follows that the principal part of $f(z)$ is $\frac{1}{z}$ and so $\mathrm{res}_{z=0}(f) = 1$. Alternatively,

$$\mathrm{res}_{z=0}(f) = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z \to 0} \frac{\sin(z) - z\cos(z) + \cos(z)}{(1-z)^2} = 1$$

Note that the first calculation also shows that $\frac{\sin z}{z^3(1-z)}$ has residue 1 at $z=0$ while $\frac{\sin z}{z^4(1-z)}$ has residue $1 - \frac{1}{3!}$ at $z=0$.

28. Cauchy's residue formula

Theorem 28.1 (Cauchy's residue formula)**.** *Let $U \subset \mathbb{C}$ be a convex open subset and suppose f is holomorphic on $U \setminus \{p_1, \dots, p_n\}$ and has a pole at each $p_i \in U$. Then for every closed curve $\gamma \subset U \setminus \{p_1, \dots, p_n\}$*

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^{n} \mathrm{res}_{z=p_i}(f) W(\gamma, p_i)$$

where $W(\gamma, p_i)$ is the winding number of γ around p_i (see Definition 14.1).

In particular, if γ is a simple closed curve oriented counterclockwise and all the poles p_i are contained inside γ then

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^n \mathrm{res}_{z=p_i}(f)$$

Proof. For each pole p_i let $S_i(z)$ denote the principal part of f at p_i. Then $S_i(z)$ is holomorphic on $U \setminus \{p_i\}$ and $f - S_i$ has a removable singularity at p_i. Therefore

$$f - \sum_{i=1}^n S_i$$

extends to a holomorphic function on U and so Cauchy's theorem for convex domains (Theorem 12.4) implies

$$\int_\gamma f(z) - \sum_{i=1}^n S_i(z)dz = 0$$

In other words,

$$\int_\gamma f(z)dz = \sum_{i=1}^n \int_\gamma S_i(z)dz$$

Now suppose $a_j^{(i)} \in \mathbb{C}$ are such that $S_i(z) = \frac{a_{-n_i}^{(i)}}{(z-p_i)^{n_i}} + \ldots + \frac{a_{-1}^{(i)}}{(z-p_i)}$. We have already seen that

$$\int_\gamma \frac{1}{(z-p)^j} dz = 0$$

for $j \neq 1$ (since $\frac{1}{(z-p)^j}$ has a primitive) and so

$$\begin{aligned} \int_\gamma f(z)dz &= \sum_{i=1}^n \int_\gamma \frac{a_{-1}^{(i)}}{(z-p_i)} dz = a_{-1}^{(i)} 2\pi i W(\gamma, p_i) \\ &= 2\pi i \sum_{i=1}^n \mathrm{res}_{z=p_i}(f) W(\gamma, p_i) \end{aligned}$$

□

Note that this generalises both Cauchy's theorem (Theorem 12.4) because if f has no poles then the right hand side is zero. It also generalises Cauchy's integral formula for derivatives (Theorem 17.1) because if f has no poles then $\frac{f(w)}{(w-z)^{n+1}}$ has residue equal $\frac{1}{n!} f^{(n)}(z)$ at z.

Lecture 7.2

29. Using the residue formula

Example 29.1. We can use Cauchy's residue formula (Theorem 28.1) to compute

$$\int_\gamma \frac{z^2+1}{z(1-z)}dz$$

for γ the circle of radius 2 centred at the origin and oriented counterclockwise. Since f has two poles in the interior of γ, namely $z=0$ and $z=1$, we just need to compute the residue's at these two poles. Since they are both poles of order 1 we have

$$\mathrm{res}_{z=0} f = \lim_{z\to 0} \frac{z^2+1}{(1-z)} = 1$$

and

$$\mathrm{res}_{z=1} f = \lim_{z\to 1} \frac{(z-1)(z^2+1)}{z(1-z)} = -2$$

Therefore,

$$\int_\gamma \frac{z^2+1}{z(1-z)}dz = 2\pi i(1-2) = -2\pi i$$

Example 29.2. Consider the integral $\int_\gamma \frac{\sin z}{z^3(z-2)}dz$ with γ the circle of radius 1 centred at the origin and parametrised counter clockwise. Since $f(z) = \frac{\sin z}{z^3(z-2)}$ has two poles at $z=0$ and $z=2$, and only one of these is contained in the interior of γ, Cauchy's residue formula implies

$$\int_\gamma f(z)dz = 2\pi i\, \mathrm{res}_{z=0}(f)$$

The binomial expansion computes the Taylor series of $\frac{1}{z-2}$ as

$$\frac{1}{z-2} = \frac{-1}{2}\frac{1}{1-(z/2)} = \frac{-1}{2}\left(1+(z/2)+(z/2)^2+\dots\right)$$

for $|z|<1$. Since $\sin z$ has Taylor expansion $z - z^3/3! + z^5/5! - \dots$ we have

$$\frac{\sin z}{z-2} = \frac{-1}{2}\left(1+(z/2)+(z/2)^2+\dots\right)\left(z - z^3/3! + z^5/5! - \dots\right)$$

To compute the residue of $f(z)$ we need to compute the z^2 term in this product, and this is

$$\frac{-1}{2}\cdot\frac{1}{2} = \frac{-1}{4}$$

Thus $\mathrm{res}_{z=0}(f) = -1/4$ and $\int_\gamma f(z)dz = 2\pi i\cdot(-1/4) = -i\pi/2$.

30. Meromorphic functions

Let $U \subset \mathbb{C}$ be an open subset. Recall that $P \subset U$ is discrete if for every $p \in P$ there exists an open neighbourhood $p \in U' \subset U$ not containing any other point of P.

Definition 30.1. Let $U \subset \mathbb{C}$ be an open set. We say that a function f is meromorphic on U if there is a discrete closed subset $P \subset U$ such that $f : U \smallsetminus P \to \mathbb{C}$ is holomorphic and f has an isolated pole at each $p \in P$.

Note P has no limit points in U, but it could have limit points on the boundary of U.

Example 30.2.
(1) If $p(z), q(z)$ are polynomials then the rational function $\frac{p(z)}{q(z)}$ is a meromorphic function on $\mathbb{C}$.
(2) More generally, if $f, g : U \to \mathbb{C}$ are holomorphic and g is not identically zero on a connected component of U then $\frac{f}{g}$ is meromorphic on U. This uses the fact that g only has isolated zeroes (Theorem 22.1).
(3) $f(z) = \frac{1}{e^{1/z}}$ is meromorphic on $\mathbb{C} \smallsetminus \{0\}$ but not on $\mathbb{C}$ (since the singularity at $z=0$ is an essential one).

31. The argument principle

Theorem 31.1 (The argument principle)**.** *Suppose that $U \subset \mathbb{C}$ is open and f is meromorphic on U. If U contains a counterclock wise oriented circle γ and its interior and f has no zeroes or poles on γ then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{f'(z)}dz = \sum_{p\in \mathrm{Zeroes}(f)} \mathrm{ord}_p(f) - \sum_{p\in \mathrm{Poles}(f)} \mathrm{ord}_p(f)$$

where $\mathrm{ord}_p(f)$ denotes the order of a zero or pole p of f and the sums run over zeroes and poles of f which are contained in the interior of γ.

Proof. By Cauchy's residue formula (Theorem 28.1) we can compute the integral by describing the poles of f'/f in terms of the zeroes and poles of f. To do this we observe that if f_1 and f_2 are meromorphic functions then the product formula for differentiation gives

$$\frac{f_1'f_2'}{f_1f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

Now suppose f has a zero of order n at $z = p$. Then $f(z) = (z-p)^n g(z)$ in a neighbourhood of p with g holomorphic (by Section 26) and $g(p) \neq 0$. The above formula gives

$$\frac{f'(z)}{f(z)} = \frac{\frac{d}{dz}(z-p)^n}{(z-p)^n} + \frac{g'(z)}{g(z)} = \frac{n}{z-p} + \frac{g'(z)}{g(z)}$$

in a neighbourhood of p. Also, since $g(p) \neq 0$ we have $\frac{g'}{g}$ holomorphic in a neighbourhood of p. It follows that f'/f has a pole of order 1 at $z = p$ and $\mathrm{res}_{z=p} f'/f = n = \mathrm{ord}_{z=p}(f)$. Similarly, if f has a pole at $z = p$ of order n then $f(z) = \frac{g(z)}{(z-p)^n}$ in a neighbourhood of p with $g(z)$ holomorphic and $g(p) \neq 0$. Thus

$$\frac{f'(z)}{f(z)} = \frac{\frac{d}{dz}(z-p)^{-n}}{(z-p)^n} + \frac{g'(z)}{g(z)} = \frac{-n}{(z-p)} + \frac{g'(z)}{g(z)}$$

Therefore, if f has a pole of order n at p then f'/f has a pole of order one at p and $\mathrm{res}_{z=p}(f'/f) = -n = -\mathrm{ord}_{z=p}(f)$.

Finally, note that if f'/f has a pole at $z = p$ then either p is a pole or zero of f, or p is a pole of f'. However, if f' has a pole at $z = p$ then f must have a pole also, because otherwise f would be holomorphic in a neighbourhood of p and so f' would be also by Cauchy's integral formula for derivatives (Theorem 17.1).

To put this all together, choose a disk D inside of U which contains γ and its interior. Cauchy's residue formula (Theorem 28.1) implies

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_p \mathrm{res}_{z=p}(f'/f)$$

By the above we know

$$\sum_{p\in \mathrm{Pole}(f'/f)} \mathrm{res}_{z=p}(f'/f) = \sum_{p\in \mathrm{Zeroes}(f)} \mathrm{ord}_p(f) - \sum_{p\in \mathrm{Poles}(f)} \mathrm{ord}_p(f)$$

and the theorem follows. □

Example 31.2. Consider the integral $\int_\gamma \frac{z^3}{z^4-1}dz$ where γ is the circle of radius 2 centred at the origin and parametrised counterclockwise. Notice that $4z^3 = \frac{d}{dz}(z^4-1)$ and z^4-1 has four zeroes at $z = 1, -1, i, -1$. Therefore, the argument principle implies

$$\int_\gamma \frac{4z^3}{z^4-1}dz = 2\pi i\left(\mathrm{ord}_1(z^4-1) + \mathrm{ord}_{-1}(z^4-1) + \mathrm{ord}_i(z^4-1) + \mathrm{ord}_{-i}(z^4-1)\right)$$

Since $z^4-1 = (z-1)(z+1)(z-i)(z+i)$ each of these zeroes is of order one. Therefore

$$\int_\gamma \frac{4z^3}{z^4-1}dz = 2\pi i \cdot 4 = 8\pi i$$

and so

$$\int_\gamma \frac{z^3}{z^4-1}dz = 2\pi i$$

Lecture 8.1

32. Rouché's theorem

Theorem 32.1 (Rouché's theorem)**.** *Suppose $U \subset \mathbb{C}$ is open and contains a closed disk $\overline{D} = \overline{D}_r(z_0)$ with boundary $\partial D = \{z \in \mathbb{C} \mid |z - z_0| = r\}$. Suppose $f, g : U \to \mathbb{C}$ are holomorphic and*

$$|f(z) - g(z)| < |f(z)|$$

for every $z \in \partial D$. Then f and g have the same number of zeroes in D (counting multiplicities) and no zeroes on ∂D.

The assertion that f and g have the same number of zeroes in D with multiplicity can be expressed as saying

$$\sum_{p \in \mathrm{Zeroes}(f)} \mathrm{ord}_p(f) = \sum_{p \in \mathrm{Zeroes}(g)} \mathrm{ord}_p(g)$$

Proof. The inequality ensures neither $g(z) = 0$ nor $f(z) = 0$ for any $z \in \partial D$. Therefore $h = \frac{g}{f}$, which is meromorphic on U, has no zeroes or poles on ∂D. The Argument principle (Theorem 31.1) therefore ensures that

$$\int_{\partial D} \frac{h'(z)}{h(z)} dz = 2\pi i \left(\sum_{p \in \mathrm{Zeroes}(g)} \mathrm{ord}_p(g) - \sum_{p \in \mathrm{Zeroes}(f)} \mathrm{ord}_p(f) \right)$$

when ∂D is traversed counter clockwise. Therefore, we will be done if we can show $\int_{\partial D} \frac{h'(z)}{h(z)} dz = 0$. To do this let γ be a parametrisation of ∂D and rewrite the inequality in the hypothesis as

$$|1 - h(\gamma(t))| = \left| 1 - \frac{g(z)}{f(z)} \right| < 1$$

for $z = \gamma(t)$. Since $h(\gamma(t)) \neq 0$ it follows that the curve $h \circ \gamma$ is contained in the open disk $D_1(1)$. Since $D_1(1)$ is convex and $\frac{1}{z}$ is holomorphic on $D_1(1)$ it follows from Theorem 12.4 that

$$0 = \int_{h \circ \gamma} \frac{1}{z} dz = \int_0^2 \pi \frac{h'(\gamma(t))}{h(\gamma(t))} dt = \int_\gamma \frac{h'(t)}{h(t)} dt$$

which finishes the proof. ☐

The following is a standard application of this result:

Example 32.2. How many roots does $f(z) = 1 + 2z + 7z^2 + 3z^5$ have in the unit disk? To solve this take $g(z) = 7z^2$. Then for z on the unit circle one has

$$|f(z) - g(z)| \leq 1 + 2 + 3 = 6 \leq 7 = |g(z)|$$

Therefore, Theorem 32 applied with $\overline{D}$ equal the closed unit disk implies that $f(z)$ and $g(z)$ have the same number of zeroes in D. Since $g(z)$ clearly has two (0, with multiplicity two) the same is true of f.

Example 32.3. Show that all the roots of $g(z) = z^6 + z + 1$ are contained in the annulus

$$\{z \in \mathbb{C} \mid \frac{1}{2} < |z| < 2\}$$

First set $f(z) = z^6$ and note that for $|z| = 2$ one has

$$|f(z) - g(z)| = |z + 1| \leq 3 < 2^6 = |f(z)|$$

Therefore Rouché's theorem implies g has the same number of zeroes as f in $D_2(0)$. In other words all 6 zeroes of g are contained in $D_2(0)$. It remains to show that g has no zeroes on $\overline{D}_{1/2}(0)$. For this compare g with $h(z) = 1$ for $|z| = 1/2$:

$$|h(z) - g(z)| = |z^6 + z| \leq (1/2)^6 + 1/2 = 1/64 + 1/2 < 1 = |h(z)|$$

Again Rouch'e's theorem implies g and h have the same number of zeroes on $D_{1/2}(0)$ and no zeroes on $\partial D_{1/2}(0)$ so $g(z) \neq 0$ for $|z| \leq 1/2$.

33. Open mapping theorem

Definition 33.1. A map of topological spaces $f : X \to Y$ is open if $U \subset X$ open implies $f(U) \subset Y$ is open.

Theorem 33.2 (Open mapping theorem)**.** *A non-constant holomorphic function $f : U \to \mathbb{C}$ on a connected open subset $U \subset \mathbb{C}$ is an open map.*

Proof. It suffices to show that $f(U) \subset \mathbb{C}$ is open because if $U' \subset U$ then we can repeat the argument with f replaced by its restriction to U'. For this we choose $w = f(p)$ for some $p \in U$ and show that

$$D_\epsilon(w) \subset \mathrm{Image}(f)$$

for some $\epsilon > 0$. Consider the function

$$F(z) = f(z) - w$$

Since f is non-constant so is F. Thus Theorem 22.1 implies p is an isolated zero of F and so there is a closed disk $\overline{D}$ around p upon which F only vanishes at p. Set

$$\epsilon = \min_{z \in \partial D} |F(z)|$$

Since $F(z) \neq 0$ for $z \in \partial D$ it follows by compactness of ∂D that $\epsilon > 0$. We can therefore choose $q \in D_\epsilon(w)$ not equal w. Define $G(z) = f(z) - q$. Then for any $z \in \partial D$ we have

$$|F(z) - G(w)| = |w - q| < \epsilon \leq |F(z)|$$

Therefore, Rouché's theorem implies F and G have the same number of zeroes in D. Since F has exactly one zero so does G. It follows that there exists $z \in D$ such that $G(z) = 0$, i.e.

$$f(z) = q$$

and so $q \in \mathrm{Image}(f)$. Since q was any point in $D_\epsilon(w)$ it follows that $D_\epsilon(w) \subset \mathrm{Image}(f)$ as required. □

Example 33.3. There exists no non-constant holomorphic function on any connected open subset $U \subset \mathbb{C}$ such that

$$\mathrm{Re}(f) = \mathrm{Im}(f)$$

If such an f existed then Image(f) would be contained inside the line $L = \{x = y\}$ in $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$. However, L is not open in $\mathbb{C}$ and therefore the Image(f) is not open in $\mathbb{C}$. This contradicts the open mapping principle.

34. Maximum modulus principle

Theorem 34.1 (Maximum modulus principle)**.** *Let $f : U \to \mathbb{C}$ be holomorphic with $U \subset \mathbb{C}$ connected and open. If f is non-constant then $|f|$ cannot obtain a maximum on U.*

Proof. Suppose $p \in U$ attains a maximum for $|f|$. Let $q = f(p)$. The open mapping theorem (Theorem 33.2) implies $f(U)$ is open and therefore contains an open disk around q. However, any open disk around q contains points whose modulus is strictly greater than q. In particular, there exists $z \in U$ with $|f(z)| > |f(p)|$ which is a contradiction. □

Lecture 9.1

35. More on essential singularities

In the last few lectures we've looked at the structure of holomorphic functions with singularities which are either removable singularities or poles. Now we'll see what can be said about the remaining kind of singularity, essential singularities.

Theorem 35.1 (Casorati–Weierstrass)**.** *Suppose that $U \subset \mathbb{C}$ is an open subset and that $f : U \setminus \{p\} \to \mathbb{C}$ is holomorphic with an essential singularity at p. Then* $\mathrm{Image}(U \setminus \{p\})$ *is dense in* $\mathbb{C}$.

Proof. Argue by contradiction. If the image is not dense then there exists $w \in \mathbb{C}$ and $\delta > 0$ such that

$$|f(z) - w| > \delta$$

for all $z \in U \setminus \{p\}$. We can therefore define a new function

$$g(z) = \frac{1}{f(z) - w}$$

which is holomorphic on $U \setminus \{p\}$ (since $f(z) - w \neq 0$) and is bounded by δ^{-1}. Riemann's removable singularity theorem (Theorem 25.1) therefore implies that g has a removable singularity at p and so can be extended to a holomorphic function $g : U \to \mathbb{C}$. If $g(p) \neq 0$ then $f(z) - w = \frac{1}{g(z)}$ is holomorphic in a neighbourhood of p which implies $f(z) - w$ (and hence f also) has a removable singularity at p. This contradicts the fact that f has an essential singularity at p. If $g(p) = 0$ then $|f(z) - w| = |\frac{1}{g(z)}| \to \infty$ as $z \to p$ and so $f(z) - w$ (and hence f also) has a pole at p. This again contradicts the fact that f has an essential singularity at p. Therefore, the $\mathrm{Image}\, U \setminus \{p\}$ must be dense. ☐

Combined with the open mapping theorem (Theorem 33.2) this shows that the image under a holomorphic function of any disc around an essential singularity is "almost" everything. In fact, this can be stated in a very strong way:

Theorem 35.2 (Great Picard's theorem)**.** *With hypotheses as in Theorem 35.1 the image of $U \setminus \{p\}$ contains every $z \in \mathbb{C}$ with at most one exception. Moreover, if $p \in$* $\mathrm{Image}\, U \setminus \{p\}$ *then $f(z) = p$ for infinitely many $z \in U \setminus \{p\}$.*

Proof. The proof of this is more difficult and we won't cover it in this course. ☐

For example, $f(z) = e^{1/z}$ attains every complex number for z in a neighbourhood of 0, except for $0 \in \mathbb{C}$.

36. Conformal mappings

Definition 36.1. Let U and V be open subsets of $\mathbb{C}$. A conformal isomorphism is a holomorphic bijection $f : U \to V$.

Example 36.2.
(1) The non-constant affine maps $f(z) = az + b$ for $a, b \in \mathbb{C}, a \neq 0$ are conformal isomorphisms of $\mathbb{C}$ with itself. The inverse of f is $g(z) = \frac{1}{a}(z - b)$.
(2) The map $f(z) = e^z$ is a conformal isomorphism of the strip
$$\{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < \pi\}$$
onto
$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$
The inverse if the complex logarithm on $\mathbb{H}$ given by $\mathrm{Log}(re^{i\theta}) = \log(r) + i\theta$ for $\theta \in [0, \pi]$ for log the real logarithm.
(3) $f(z) = -\frac{1}{2}(z + z^{-1})$ is a conformal isomorphism of $\mathbb{D}^+ = \{z \in \mathbb{D} \mid \mathrm{Im}\, z > 0\}$ onto $\mathbb{H}$.
(4) The map $F(z) = \frac{i-z}{i+z}$ is a conformal isomorphism $\mathbb{H} \to \mathbb{D}$ with inverse is given by $G(w) = i\frac{1-w}{1+w}$.

Lecture 9.2

Proposition 36.3. *Suppose $f : U \to V$ is a conformal bijection. Then $f'(z) \neq 0$ for all $z \in U$ and the set theoretic inverse $f^{-1} : V \to U$ is holomorphic with*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$$

Proof. Suppose that $f'(p) = 0$ for some $p \in U$. Then f has Taylor expansion

$$f(z) = a_0 + \sum_{n \geq m} a_n (z-p)^n$$

around p with $m \geq 2$ and $a_m \neq 0$. Therefore $F(z) := f(z) - a_0$ has an isolated zero of order $m \geq 2$ at p. Since the zero is isolated we can find $\delta > 0$ such that F has no other zero in $\overline{D}_\delta(p)$. We can also choose δ so that $f'(z) \neq 0$ on $D_\delta(p)$ (if this were the case then all higher derivatives of f would vanish on $D_\delta(p)$ and so f would be constant, contradicting the fact that $f$is a bijection). Let

$$\epsilon = \min_{z \in \partial D_\delta(p)} |F(z)|$$

and consider a non-zero $q \in D_\epsilon(0)$. We then apply Rouché's theorem to $G(z) = F(z) - q$ and deduce, since

$$|q| = |G(z) - F(z)| < |F(z)|$$

for all $z \in \partial D_\delta(p)$, that G and F have the same number of zeroes in $D_\delta(p)$. Therefore G has m zeroes in this disc. It these zeroes were repeated then $G(z) = (z-w)^m G_1(z)$ for $w \in D_\delta(p)$ and $G_1(z)$ holomorphic. But then $G'(z) = m(z-w)^{m-1} G_1(z) + (z-w)^m G'(z)$ and so $G'(w) = 0$. However, $G'(w) = f'(w)$ and we assumed $f'(z) \neq 0$ for all $z \in D_\delta(p)$. It follows that the zeroes of G are all distinct which contradicts the assumption that f, and hence G also, is injective.

To finish we have to show that f^{-1} is holomorphic. The open mapping theorem (Theorem 33.2) applied to f gives that if $U' \subset U$ is open then

$$(f^{-1})^{-1}(U) = f(U)$$

is open also. Therefore f^{-1} is continuous and so if $w = f(z)$ and $w_0 = f(z_0)$ then if $w \to w_0$ then $z \to z_0$. Since

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}$$

it follows that, as $w \to w_0$, this converges to $1/f'(z_0)$. As $f'(z_0) \neq 0$ this limit exists and so f^{-1} is holomorphic and $(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$. □

37. Schwarz's lemma

Theorem 37.1 (Schwarz's Lemma)**.** *Recall $\mathbb{D} = D_1(0)$. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then*

(1) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
(2) If $|f(p)| = |p|$ for some $p \in \mathbb{D} \setminus \{0\}$ then $f(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$. In other words f is a rotation of $\mathbb{D}$.
(3) $|f'(0)| \leq 1$ and if $|f'(0)| = 1$ then f is also a rotation of $\mathbb{D}$.

Proof. Define

$$g(z) = \begin{cases} \frac{f(z)}{z} = \frac{f(z) - f(0)}{z - 0} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Then g is holomorphic on $\mathbb{D} \setminus \{0\}$ and continuous at 0. Riemann's removable singularity theorem (Theorem 25.1) implies g is holomorphic at $z = 0$. For any $0 < r < 1$ the maximum modulus principle (Theorem 34.1) ensures

$$\max_{z \in \overline{D_r(0)}} |g(z)| = \max_{z \in \partial D_r(0)} \left| \frac{f(z)}{z} \right|$$

Therefore $\max_{z \in \overline{D_r(0)}} |g(z)| \leq \frac{1}{r}$. Letting $r \to 1$ gives that

$$\sup_{z \in \mathbb{D}} |g(z)| \leq 1$$

This shows that

- If $z \in \mathbb{D} \setminus \{0\}$ then $\frac{|f(z)|}{|z|} \leq 1$ which proves (1).
- $|g(0)| = |f'(0)| \leq 1$ which proves the first part of (3).
- If for some $p \in \mathbb{D}$ one has $|g(p)| = 1$ then $|g|$ achieves its maximum in the interior of $\mathbb{D}$ so the maximum modulus principle (Theorem 34.1) implies g is constant. As $|g(p)| = 1$ we must have $g(z) = e^{i\theta}$ for all $z \in \mathbb{D}$ and some $\theta \in [0, 2\pi)$. This implies
$$f(z) = e^{i\theta} z$$
for all $z \in \mathbb{D} \setminus \{p\}$. If $p \in \mathbb{D} \setminus \{0\}$ this proves (2). If $p = 0$ then this proves the last part of (3).

□

Lecture 10.1

38. Conformal automorphisms of the disk

Definition 38.1. If $U \subset \mathbb{C}$ is open then a conformal isomorphism $f : U \to U$ is called a conformal automorphism.

Note that the set of conformal automorphisms of U form a group under composition.

Proposition 38.2. *If $\alpha \in \mathbb{D}$ then*

$$f_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

is a conformal automorphism of $\mathbb{D}$.

Proof. We see that f_α is holomorphic on $\mathbb{C} \setminus \{\frac{1}{\overline{\alpha}}\}$. Since $|\alpha| = |\overline{\alpha}|$ if $|\alpha| < 1$ then $\frac{1}{|\overline{\alpha}|} > 1$. In particular, f_α is holomorphic on $\mathbb{D}$.

Next, we have to show that $f_\alpha(z) \in \mathbb{D}$ for all $z \in \mathbb{D}$. For this note that f_α is defined on $\partial\mathbb{D}$ and

$$f_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{1 - \overline{\alpha} e^{i\theta}} = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \overline{\alpha})} = \frac{-1}{e^{i\theta}} \frac{w}{\overline{w}}$$

for $w = e^{i\theta} - \alpha$. Therefore $|f_\alpha(e^{i\theta})| = 1$ for all $e^{i\theta}$ and so, since f_α is not constant, the maximum modulus principle implies $|f_\alpha(z)| < 1$ for all $z \in \mathbb{D}$. In other words $f_\alpha(z) \in \mathbb{D}$ for all $z \in \mathbb{D}$.

Finally, one shows that f_α is its own inverse. Indeed

$$f_\alpha \circ f_\alpha(z) = \frac{\alpha - \frac{\alpha - z}{1 - \overline{\alpha} z}}{1 - \overline{\alpha} \frac{\alpha - z}{1 - \overline{\alpha} z}} = \frac{z - \alpha\overline{\alpha} z}{1 - \alpha\overline{\alpha}} = z$$

In particular, it follows that f_α is a bijection and hence a conformal isomorphism. □

Theorem 38.3. *Any conformal automorphism of $\mathbb{D}$ has the form*

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z}$$

for some $\theta \in [0, 2\pi)$ and $\alpha \in \mathbb{D}$.

Proof. Let $\alpha = f^{-1}(0)$. Since $f_\alpha(0) = \alpha$ we see that if

$$g = f \circ f_\alpha$$

then $g(0) = f(f_\alpha(0)) = f(\alpha) = f(f^{-1}(0))$. Schwarz's Lemma then implies $|g(z)| \leq |z|$. Since g is a composition of two conformal automorphisms it is also a conformal automorphism. Therefore so is g^{-1}. Since $g^{-1}(0) = 0$ Schwarz's lemma also implies $|g^{-1}(z)| \leq |z|$. Therefore

$$|g(z)| \leq |z| = |g^{-1}(g(z))| \leq |g(z)|$$

and so $|g(z)| = |z|$ for all $z \in \mathbb{D}$. Therefore Schwarz's lemma implies $g(z) = e^{i\theta} z$ and so

$$f(z) = f \circ \underbrace{f_\alpha(z) \circ f_\alpha(z)}_{=\mathrm{Id}} = g \circ f_\alpha(z) = e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z}$$

□

39. Riemann mapping theorem

Definition 39.1. We say that two curves $\gamma_0, \gamma_1 : [a, b] \to U$ in an open set $U \subset \mathbb{C}$ are homotopic if

$$\gamma_0(a) = \gamma_1(a) = \alpha, \qquad \gamma_0(b) = \gamma_1(b) = \beta$$

and there exists curves $\gamma_s : [a, b] \to U$ for all $s \in [0, 1]$ so that each $\gamma_s(a) = \alpha$, $\gamma_s(b) = \beta$ for each s so that

$$[a, b] \times [0, 1] \to U$$

given by $(t, s) \mapsto \gamma_s(t)$ is continuous.

Definition 39.2. A set is simply connected if it is connected and any two paths with the same end points are homotopic.

This is a more refined version of our notion of convexity, and in fact one can prove an improved version of Cauchy's theorem (Theorem 12.4) where U is only assumed to be simply connected.

Example 39.3. Any convex set U if simply connected. Indeed, if γ_0 and γ_1 are two contours in U then set

$$\gamma_s(t) = \gamma_0(t)(1-s) + \gamma_1(t)s$$

For any t and $s \in [0,1]$ we see that $\gamma_s(t)$ lies on the line between $\gamma_0(t)$ and $\gamma_1(t)$. Therefore, $\gamma_s(t) \in U$ for any s and t.

Theorem 39.4. *Let $U \subsetneq \mathbb{C}$ be a simply connected open set. Then there exists a conformal isomorphism $f : U \to \mathbb{D}$.*

Sketch of proof. First one constructs an injective holomorphic map $\psi : U \to \mathbb{D}$. For this one takes $w_0 \notin U$ (which is possible since $U \neq \mathbb{C}$) and considers the squareroot f of the function $z \mapsto z - w_0$ on U. Thus $f(z)^2 = z - w_0$ and that such a square root exists uses the fact that U is simply connected. Clearly f is injective because $f(z_1) = f(z_2)$ implies $z_1 - w_0 = z_2 - w_0$. To produce an injective function taking values in $\mathbb{D}$ use the open mapping theorem (Theorem 33.2) to find a disk $D_r(a) \subset f(U)$ with $0 < r < |a|$ (so $a \neq 0$). Notice that if $f(z_1) = -f(z_2)$ then we also have $z_1 = z_2$ and so $f(z) \neq -a$ for any $z \in U$. In particular

$$\psi(z) = \frac{r}{f(z) + a}$$

is holomorphic on U. We also have $|\psi(z)| < 1$ and, since f is injective, so is ψ. In particular, the set

$$\mathcal{F} = \{\psi : U \to \mathbb{D} \mid \psi \text{ is injective and holomorphic}\}$$

is non-empty.

The second step is to show that if $\psi \in \mathcal{F}$ with $\psi(U) \neq \mathbb{D}$ then for each $z_0 \in U$ there exists $\psi_1 \in \mathcal{F}$ with

$$|\psi_1(z_0)| \geq |\psi'(z_0)|$$

This is done by using the automorphisms $f_\alpha = \frac{z-\alpha}{1-\overline{\alpha}z}$. Indeed, if $\alpha \in \mathbb{D} \setminus \psi(U)$ then consider $g : U \to \mathbb{D}$ with

$$g(z)^2 = f_\alpha \circ \psi$$

(again the simply connectedness of U is used to take the square root of the function). Then $g \in \mathcal{F}$ and one takes $\psi_1 = f_\beta \circ g$ for $\beta = g(z_0)$. The desired inequality is then a consequence of Schwarz's lemma.

Finally, one fixed $z_0 \in U$ and sets

$$\eta = \sup_{\psi \in \mathcal{F}} |\psi'(z_0)|$$

To finish the proof it suffices to find $\psi \in \mathcal{F}$ for which $|\psi'(z_0)| = \eta$ because the previous paragraph implies there can not then exist an element of $\mathbb{D} \setminus \psi(U)$.

$\square$

Lecture 10.2

40. Constructing entire functions with prescribed zeroes

Definition 40.1. If a sequence $a_n \in \mathbb{C}$ for $n \geq 1$ is given then we say that the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges if the limit

$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n)$$

of partial products exists.

Proposition 40.2. *If the series $\sum_{n=1}^{\infty}|a_n|$ converges (i.e. $\sum_{n=1}^{\infty}a_n$ is absolutely convergent) then $\prod_{n=1}^{\infty}(1+a_n)$ converges. Additionally, the product converges to zero if and only if $1+a_n=0$ for some n.*

Proof. If $\sum_{n=1}^{\infty}|a_n|$ converges then there exists an $N>0$ such that $|a_n|<1/2$ for $n\geq N$. It follows that

$$\log(1+a_n) := \sum_{m=1}^{\infty}(-1)^{m-1}\frac{a_n^m}{m}$$

converges for $n>N$ (because the radius of convergence of this power series is 1) and one has $1+a_n = e^{\log(1+a_n)}$. This means for $M>N$ we can write

$$P_M = \prod_{n=1}^{M}(1+a_n) = C\prod_{n=N}^{M}e^{\log(1+a_n)} = Ce^{B_M}$$

where $B_M=\sum_{n=N}^{M}b_n$ with $b_n=\log(1+a_n)$ and $C=\prod_{n=1}^{N-1}(1+a_n)$. Note that if $|z|\leq 1/2$ then

$$|\log(1+z)| \leq \sum_{n=1}^{\infty}\frac{|z^n|}{n} \leq |z|\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\right) = \frac{|z|}{1-1/2} = 2|z|$$

and so $|b_n|\leq 2|a_n|$. Therefore the sequence B_M converges to a complex number B. Since the exponential function is continuous it follows that $e^{B_M}\to e^B$. Therefore the sequence P_M converges.

Finally note that if $1+a_n\neq 0$ for all n then C is non-zero and the limit of the partial products is Ce^B is also non-zero since $e^B\neq 0$. □

Proposition 40.3. *Suppose that $F_n : U\to\mathbb{C}$ is a sequence of holomorphic functions on an open subset $U\subset\mathbb{C}$. If there exists $c_n>0$ such that $\sum_{n=1}^{\infty}|c_n|$ converges and*

$$|F_n(z)-1| < c_n \qquad \text{for all } z\in U$$

then

(1) The product $\prod_{n=1}^{\infty}F_n(z)$ converges uniformly in U to a holomorphic function $F(z)$.
(2) If $F_n(z)\neq 0$ for any $z\in U$ and $n\geq 1$ then

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty}\frac{F_n'(z)}{F_n(z)}$$

Proof. For the first part write $F_n(z)=1+a_n(z)$ with $|a_n(z)|\leq c_n$. Then the previous argument shows there exists $N>0$ such that for $M>N$ we have

$$P_M(z)\prod_{n=1}^{M}(1+a_n(z)) = C(z)e^{B_M(z)}$$

with $C(z)=\prod_{n=1}^{N-1}(1+a_n(z))$ and $B_M(z)=\sum_{n=N}^{M}b_n(z)$ with $|b_n(z)|\leq 2|a_n(z)|\leq 2c_n$. This shows that $\sum_{n=N}^{M}b_n(z)$ converges uniformly in z. Since the integer N depends only on the c_n it is also independent of z. It follows that the $P_M(z)$ converges uniformly on U. Since the uniform limit of holomorphic functions is holomorphic (Theorem 20.2) it follows that $\prod_{n=1}^{\infty}F_n(z)$ is also holomorphic.

The second part uses that if a sequence of holomorphic functions f_n converges uniformly to f then the sequence of derivatives f_n' converges uniformly to f' on any disk in U. This implies that

$$\frac{F'(z)}{F(z)} = \lim_{n\to\infty}\frac{\frac{d}{dz}\prod_{n=1}^{N}F_n(z)}{\prod_{n=1}^{N}F_n(z)}$$

But the product rule gives

$$\frac{d}{dz}\prod_{n=1}^{N}F_n(z)=\sum_{k=1}^{N}F_k'(z)\left(\prod_{n\neq k}F_n(z)\right)$$

and so

$$\frac{F'(z)}{F(z)}=\sum_{k=1}^{\infty}\frac{F_k'(z)}{F_k(z)}$$

□

41. Weierstrass infinite products

Theorem 41.1. *(Weierstrass infinite products) Suppose that $a_n\in\mathbb{C}$ is a sequence with $|a_n|\to\infty$ as $n\to\infty$. Then there exists an entire function $f:\mathbb{C}\to\mathbb{C}$ such that f has zeroes only at each a_n, and these zeroes are of order n_i. Moreover, any other such function with this property has the form $f(z)e^{g(z)}$ for an entire function $g(z)$.*

Note here the sequence a_n can have repetitions, and in this case we say f has multiple zeroes at the same a_n if the order of the zero equals the multiplicity of a_n in the sequence.

Sketch of proof. The naive guess to construct f would be as $\prod_{n=1}^{\infty}(1-z/a_n)$. However, one cannot guarantee this product converges. Instead, takes

$$f(z)=z^m\prod_{n=1}^{\infty}E_n(z/a_m)$$

where m is the multiplicity of 0 in the sequence a_n and we define

$$E_k(z)=\begin{cases}(1-z) & \text{if } k=0\\ (1-z)e^{z+z^2/2+\dots z^k/k} & \text{if } k\geq 1\end{cases}$$

To check convergence of this infinite product one argues in a disk of radius $R>0$. One shows that if $|z|\leq 1/2$ then

$$|1-E_k(z)|\leq c|z|^{k+1}$$

for a constant c not depending on k. In particular, if $|a_n|>2R$ then

$$|1-E_n(z/a_m)|\leq\frac{c}{2^{n+1}}$$

and so the product converges uniformly. □

Example 41.2. A key example of this construction is the formula

$$\frac{\sin\pi z}{\pi}=z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

To show this first consider the function on $\mathbb{C}\setminus\mathbb{Z}$ defined by

$$F(z)=\sum_{n=-\infty}^{\infty}\frac{1}{z+n}=\lim_{N\to\infty}\sum_{|n|\leq N}\frac{1}{z+n}=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

This function satisfies the properties:

(1) $F(z+1)=F(z)$
(2) $F(z)=\frac{1}{z}+F_0(z)$ with $F_0(z)$ holomorphic in a neighbourhood of zero.
(3) $F(z)$ has simple poles at the integers and no other singularities.

These three properties are also satisfied by the function $\pi\cot(\pi z)=\pi\frac{\cos\pi z}{\sin\pi z}$ and so

$$\Delta(z)=\pi\cot(\pi z)-F(z)$$

is periodic (i.e. $\Delta(z+1)=\Delta(z)$) and has a removable singularity at the origin (and so also at all the integers). It follows that Δ is entire. One then shows (by arguments that we don't give here) that Δ is bounded on $\mathbb{C}$. Therefore Liouville's theorem (Theorem 19.2) gives

$$\pi\cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

Now we can use this to prove our product formula for sin. Set $G(z) = \frac{\sin \pi z}{\pi}$ and $P(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$. Note $P(z)$ converges because the series $\sum_{n=1}^{\infty} 1/n^2$ converges. We also have:

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

by Proposition 40.3. Since $G'(z)/G(z) = \pi \cot \pi z$ it follows that $G'(z)/G(z) = P'(z)/P(z)$ and so

$$\left(\frac{G(z)}{P(z)}\right)' = \frac{P(z)}{G(z)} \left(\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}\right) = 0$$

Thus $P(z) = cG(z)$ for a constant c and dividing by π and letting $z \to 0$ shows $c = 1$.