CYCLES RELATIONS IN THE AFFINE GRASSMANNIAN AND APPLICATIONS TO BREUIL–MÉZARD FOR G-CRYSTALLINE REPRESENTATIONS

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Abstract. For a split reductive group G we realise identities in the Grothendieck group of $\widehat{G}$-representations in terms of cycle relations between certain closed subschemes inside the affine grassmannian. These closed subschemes are obtained as a degeneration of e-fold products of flag varieties and, under a bound on the Hodge type, we relate the geometry of these degenerations to that of moduli spaces of G-valued crystalline representations of $\mathrm{Gal}(\overline{K}/K)$ for $K/\mathbb{Q}_p$ a finite extension with ramification degree e. By transferring the aforementioned cycle relations to these moduli spaces we deduce one direction of the Breuil–Mézard conjecture for G-valued crystalline representations with small Hodge type.

Contents

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1. Introduction

The goal of this paper is to prove new results towards the Breuil–Mézard conjecture for crystalline representations valued in a connected split reductive group G. This open conjecture is a combinatorial shadow of the expected p-adic Langlands correspondence and describes multiplicities of irreducible components inside moduli spaces of p-adic Galois representations. We refer to the introduction of [Bar21] or [EG23, §1.7] for more details, at least when $G = \mathrm{GL}_n$. To achieve our goal we describe new structures in the affine grassmannian which exhibit Breuil–Mézard phenomena, and relate these to moduli of Galois representations. When $G = \mathrm{GL}_2$ these results were proven in [Bar21], and what we do here extends these techniques to general G.

There are two main theorems we prove. The first is purely algebro-geometric and describes an analogue of the Breuil–Mézard conjecture for certain closed subschemes inside the affine grassmannian. For this we fix a split reductive group G, together with a choice of maximal torus and Borel $T \subset B$ and write $\widehat{G}$ for the dual group. We let $\mathrm{Gr}_{G,\mathbb{F}}$ denote the associated affine grassmannian over a field $\mathbb{F}$ and, for an integer $e \geq 1$ and any e-tuple of dominant cocharacters $\mu = (\mu_1, \ldots, \mu_e)$ of G, we define closed subschemes $M_{\mu,\mathbb{F}} \subset \mathrm{Gr}_{G,\mathbb{F}}$ as degenerations of an e-fold product of flag varieties $G/P_{\mu_1} \times \ldots \times G/P_{\mu_e}$ (see the first bullet point after Theorem 1.1 for more details). We then show that the geometry of these $M_{\mu,\mathbb{F}}$ as μ varies can be described in terms of the representation theory of e-fold tensor products of the $\widehat{G}$-representations $W(\mu_i)$ (for any dominant cocharacter λ of G we write $W(\lambda)$ for the associated Weyl module, viewed as an algebraic representation of $\widehat{G}$). More precisely, we prove:

Theorem 1.1. *Assume G admits a twisting element ρ, i.e. a cocharacter pairing to 1 with all simple roots of G. Then, for any tuple $\mu = (\mu_1, \ldots, \mu_e)$ of strictly dominant cocharacters of G (i.e. $\mu_i - \rho$ is dominant) satisfying*

$$\sum_{i=1}^{e} \langle \alpha^\vee, \mu_i \rangle \leq \mathrm{char}\, \mathbb{F} + e - 1$$

for all roots $\alpha^\vee$ of G (when $\mathrm{char}\, \mathbb{F} = 0$ this condition is not needed) one has identities of $e \dim G/B$-dimensional cycles

$$[M_{\mu,\mathbb{F}}] = \sum_{\lambda} m_\lambda [M_{(\lambda+\rho,\rho,\ldots,\rho),\mathbb{F}}]$$

where m_λ denotes the multiplicity of $W(\lambda)$ inside $\bigotimes_{i=1}^{e} W(\mu_i - \rho)$. Furthermore, each $M_{(\lambda+\rho,\rho,\ldots,\rho),\mathbb{F}}$ appearing in this sum is irreducible and generically reduced.

A twisting element ρ need not always exist (e.g. if $G = \mathrm{SL}_2$) but will whenever $\widehat{G}$ is semi-simple and simply connected, or has simply connected derived subgroup (e.g. if $G = \mathrm{GL}_n$). Twisting elements can also always be found after replacing $\widehat{G}$ by a $\mathbb{G}_m$-extension [BG14, §5]. When $G = \mathrm{GL}_n$ the cocharacters μ_i identify with n-tuples $(\mu_{i,1}, \ldots, \mu_{i,n})$ of integers via $\mu_i(t) = \mathrm{diag}(t^{\mu_{i,1}}, \ldots, t^{\mu_{i,n}})$. Under this identification we can take $\rho = (n-1, n-2, \ldots, 1, 0)$. Then μ_i being strictly dominant is equivalent to asking that $x_i - x_{i+1} \geq 1$ for each i and the bounds in Theorem 1.1 are equivalent to asking that

$$\sum_{i=1}^{e} (\mu_{i,1} - \mu_{i,n}) \leq \mathrm{char}\, \mathbb{F} + e - 1$$

The following three points identify the crucial inputs into the proof of Theorem 1.1:

- In order to construct the degenerations $M_{\mu,\mathbb{F}}$ we choose a discrete valuation ring $\mathcal{O}$ with residue field $\mathbb{F}$ and an e-tuple of pairwise distinct $\pi_1,\dots,\pi_e$ in the maximal ideal of $\mathcal{O}$. Viewing G as a group over $\operatorname{Spec}\mathcal{O}$ we use $\pi_1,\dots,\pi_e$ to extend $\mathrm{Gr}_{G,\mathbb{F}}$ to and ind-$\mathcal{O}$-scheme Gr_G (as a specialisation of the Beilinson–Drinfeld grassmannian over $\mathbb{A}^e_{\mathbb{Z}}$) with generic fibre an e-fold product of $\mathrm{Gr}_{G,\operatorname{Frac}\mathcal{O}}$'s. Any tuple $\mu=(\mu_1,\dots,\mu_e)$ then determines a closed immersion
$$G/P_{\mu_1}\times\dots\times G/P_{\mu_e}\hookrightarrow \mathrm{Gr}_G\otimes_{\mathcal{O}}\operatorname{Frac}\mathcal{O}$$
for P_{μ_i} the associated parabolic subgroup. We set M_μ equal the closure of this embedding inside Gr_G and $M_{\mu,\mathbb{F}}$ the fibre over $\operatorname{Spec}\mathbb{F}$.
- Next we establish cycle identities
$$[M_{(\mu_1+\rho,\dots,\mu_e+\rho)}\otimes_{\mathcal{O}}\mathbb{F}]=\sum_\lambda n_\lambda[M_{(\lambda+\rho,\rho,\dots,\rho)}\otimes_{\mathcal{O}}\mathbb{F}] \tag{1.2}$$
for $n_\lambda\in\mathbb{Z}_{\geq 0}$, a priori, with no representation theoretic interpretation. This is essentially a topological calculation and is achieved by giving an explicit moduli description of a closed subscheme in Gr_G approximating M_μ in the sense that
$$M_{\mu,\mathbb{F}}\subset \mathrm{Gr}^{\nabla}_{\leq\mu}$$
and that the top dimensional irreducible components of $\mathrm{Gr}^{\nabla}_{\leq\mu}\otimes_{\mathcal{O}}\mathbb{F}$ generically identify with the $M_{(\lambda+\rho,\rho,\dots,\rho),\mathbb{F}}$ for dominant λ with $\lambda+e\rho\leq\mu_1+\dots+\mu_e$. The moduli description of $\mathrm{Gr}^{\nabla}_{\leq\mu}$ is primarily Lie theoretic, and can be viewed as either an infinitesimal version of being fixed by the loop rotation, or as an incarnation of Griffiths transversality for Breuil–Kisin modules. It is in these calculations that the restriction on $\operatorname{char}\mathbb{F}$ and the strict dominance of the μ_i play a crucial role.
- To finish the proof it remains to identify the n_λ's with the representation theoretic multiplicities m_λ. For this we consider the G-equivariant ample line bundle $\mathcal{L}$ on Gr_G obtained by pulling back the determinant bundle along the adjoint representation. The restriction of $\mathcal{L}$ to $M_{(\mu_1+\rho,\dots,\mu_e+\rho)}\otimes_{\mathcal{O}}\operatorname{Frac}\mathcal{O}$ can be expressed explicitly as a product of equivariant line bundles on flag varieties. Using ampleness of $\mathcal{L}$ and flatness of $M_{(\mu_1+\rho,\dots,\mu_e+\rho)}$ over $\mathcal{O}$ we are therefore able to identify, for sufficiently large n,
$$H^0(M_{(\mu_1+\rho,\dots,\mu_e+\rho)}\otimes_{\mathcal{O}}\mathbb{F},\mathcal{L}^{\otimes n})=\bigotimes_{i=1}^e W(np(\mu_i+\rho))$$
as G-representations over $\mathbb{F}$. Here $p(\eta)=\sum_{\alpha^\vee}\langle\alpha^\vee,\eta\rangle\alpha^\vee$ is the homomorphism from cocharacters to characters induced by the Killing form. In Section 10 we show that if X is a T-equivariant scheme admitting an equivariant ample line bundle $\mathcal{L}_X$ then any identity of cycles between T-equivariant closed subschemes induces an asymptotic formula between the global sections of high powers of $\mathcal{L}_X$ inside the Grothendieck group of T-representations. Applying this to (1.2) and $\mathcal{L}$ produces a formula involving the n_λ which asymptotically relates $[\bigotimes_{i=1}^e W(np(\mu_i+\rho))]$ and $[W(np(\lambda+\rho))\otimes W(np(\rho))^{\otimes e-1}]$, in the sense that an appropriate difference is polynomial in n of degree $<e\dim G/B$. In Sections 12 and 13 we show,

using elementary manipulations with the Weyl character formula, that such asymptotic relations can only occur if $m_\lambda = n_\lambda$, proving Theorem 1.1.

Our second main theorem specialises to the case of residue characteristic p, and proves new instances of one direction of the Breuil–Mézard conjecture (namely that Galois multiplicities are $\leq$ automorphic multiplicities). It is deduced from Theorem 1.1 by relating the geometry of the M_μ's to that of moduli spaces of crystalline representations.

Theorem 1.3. *Continue to assume G admits a twisting element ρ and let $K/\mathbb{Q}_p$ be a finite extension with ramification degree e. Let $R_{\overline{\rho}}^{\square,\mathrm{cr},\mu}$ denote the framed deformation ring of a fixed $\overline{\rho} : G_K \to G(\overline{\mathbb{F}}_p)$ classifying G-valued crystalline representations with Hodge type $\mu = (\mu_\kappa)_{\kappa : K \to \overline{\mathbb{Q}}_p}$. Suppose further that each μ_κ is strictly dominant and, for each $\kappa_0 : k \to \overline{\mathbb{F}}_p$,*

$$\sum_{\kappa|_k = \kappa_0} \langle \alpha^\vee, \mu_\kappa \rangle \leq p$$

for all roots $\alpha^\vee$ of G. Then, as $\dim G + [K : \mathbb{Q}_p] \dim G/B$-dimensional cycles,

$$[R_{\overline{\rho}}^{\square,\mathrm{cr},\mu} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p] \leq \sum_\lambda m_\lambda [R_{\overline{\rho}}^{\square,\mathrm{cr},(\lambda+\rho,\rho,\ldots,\rho)} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p]$$

where again m_λ denotes the multiplicity of the Weyl module $W(\lambda)$ of highest weight λ inside $\bigotimes_{i=1}^e W(\mu_i - \rho)$.

When $G = \mathrm{GL}_n$ we can again identify the μ_κ with n-tuples of integers $(\mu_{\kappa,1}, \ldots, \mu_{\kappa,n})$ so that $\rho = (n-1, n-2, \ldots, 1, 0)$. Then the bound on the μ_κ's is equivalent to asking that

$$\sum_{\kappa|_k = \kappa_0} (\mu_{\kappa,1} - \mu_{\kappa,n}) \leq p$$

In particular, we see that the theorem, roughly speaking, accesses Hodge types contained in the interval $[0, p/e]$. Note that since we also ask each μ_κ to be strictly dominant we have $\mu_{\kappa,1} - \mu_{\kappa,n} \geq n-1$ and so μ as in Theorem 1.3 will only exist when $e(n-1) \leq p$.

The crux of Theorem 1.3's proof lies in connecting crystalline representations to the $M_{\mu,\mathbb{F}}$. This passage is achieved via the intermediary of Breuil–Kisin modules associated to crystalline representations. To explain this we assume, for notational simplicity, that K is totally ramified over $\mathbb{Q}_p$ and let $\mathcal{O}$ be the ring of integers in a finite extension containing the Galois closure of K and with residue field $\mathbb{F}$. Let $\mathfrak{M}$ denote the Breuil–Kisin module associated to a crystalline representation valued in $G(\mathcal{O})$. This is a G-torsor on $\operatorname{Spec} \mathcal{O}[[u]]$ equipped with an isomorphism $\varphi^*\mathfrak{M}[\frac{1}{E(u)}] \cong \mathfrak{M}[\frac{1}{E(u)}]$, where φ is given by $u \mapsto u^p$ and $E(u)$ is the minimal polynomial of a fixed choice of uniformiser $\pi \in K$. Any trivialisation ι of $\mathfrak{M}$ over $\operatorname{Spec} \mathcal{O}[[u]]$ produces an $\mathcal{O}$-valued point $\Psi(\mathfrak{M}, \iota) \in \mathrm{Gr}_G$ describing the relative position of $\mathfrak{M}$ and $\varphi^*\mathfrak{M}$. We prove that if the Hodge type μ of the crystalline representation satisfies the bound from Theorem 1.3 then $\Psi(\mathfrak{M}, \iota) \otimes_{\mathcal{O}} \mathbb{F} \in M_{\mu,\mathbb{F}}$. The following describes the central idea:

- Consideration of Kisin's original construction of $\mathfrak{M}$ from [Kis06] shows that, without any bound on μ, one has

$$X \cdot \Psi(\mathfrak{M}, \iota)[\tfrac{1}{p}] \in M_\mu[\tfrac{1}{p}]$$

for $X \in G(\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}])$ the automorphism inducing a Frobenius equivariant identification $\mathfrak{M} \otimes \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}] \cong D \otimes \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$. Here D is the filtered φ-module associated to the crystalline representation, $\mathcal{O}^{\mathrm{rig}}$ is the ring of power series convergent on the open unit disk over $\mathrm{Frac}\,\mathcal{O}$, and $\lambda = \prod_{n \geq 0} \varphi^n(\frac{E(u)}{E(0)})$.

- While X will almost never be defined integrally, calculations of [Liu15] and [GLS14] bound the order of $\frac{1}{p}$ in the coefficients of X (more precisely, they bound the coefficients of the monodromy operator from which X can be recovered). Using this we are able to show that a truncation X^{trun} of X modulo sufficient high powers of $E(u)$ (depending upon μ) is such that X^{trun} is integral, $\equiv 1$ modulo the maximal ideal of $\mathcal{O}$, and still satisfies
$$X^{\mathrm{trun}} \cdot \Psi(\mathfrak{M}, \iota)[\tfrac{1}{p}] \in M_\mu[\tfrac{1}{p}]$$
Thus $\Psi(\mathfrak{M}, \iota) \otimes_{\mathcal{O}} \mathbb{F} \in M_{\mu,\mathbb{F}}$ as desired.

More generally this construction works whenever the crystalline representation is valued in $G(A)$ for A any finite flat $\mathcal{O}$-algebra for which the associated Breuil–Kisin module is a G-torsor on $\mathrm{Spec}\,A[[u]]$ (which is not automatic). As a consequence, if one considers the standard diagram (whose construction goes back to [Kis09b] and [PR09])

$$\begin{tikzcd} & & \widetilde{Y} \arrow[dl, "\Gamma"'] \arrow[dr, "\Psi"] & \\ X & Y \arrow[l, "\Theta"] & & \mathrm{Gr}_G \end{tikzcd}$$

in which X denotes an appropriate moduli space of crystalline Galois representations, Y a moduli space of Breuil–Kisin modules associated to crystalline Galois representations, and $\widetilde{Y}$ a rigidification of Y classifying an additional choice of trivialisation, then the restriction of Ψ to the fibre over $\mathrm{Spec}\,\mathbb{F}$ of the closed locus $\widetilde{Y}^\mu \subset \widetilde{Y}$ of Breuil–Kisin modules of Hodge type μ factors through $M_{\mu,\mathbb{F}}$. An additional (but much simpler) argument shows that, under the bound on μ, the morphism Ψ is formally smooth over $M_{\mu,\mathbb{F}}$. As a result the cycle identities in Theorem 1.1 can be pulled back along Ψ, descended along Γ, and then pushed forward along the proper morphism Θ. Since we only know that the preimage of $M_{\mu,\mathbb{F}}$ contains $\widetilde{Y}^\mu \otimes \mathbb{F}$ this process produces an identity of cycles
$$[X_0^\mu] = \sum_\lambda n_\lambda [X_0^{(\lambda+\rho,\rho,\ldots,\rho)}]$$
inside $X \otimes \mathbb{F}$ with $[X^\mu \otimes \mathbb{F}] \leq [X_0^\mu]$ for $X^\mu \subset X$ the locus of crystalline representations of Hodge type μ. However, by the last part of Theorem 1.1, and the fact that over each $M_{\mu,\mathbb{F}}$ the morphism Ψ is smooth with irreducible fibres, we can additionally show that $X_0^{(\lambda+\rho,\rho,\ldots,\rho)}$ is irreducible and generically reduced. Thus $[X_0^{(\lambda+\rho,\rho,\ldots,\rho)}] = [X^{(\lambda+\rho,\rho,\ldots,\rho)} \otimes \mathbb{F}]$. This gives the inequality in Theorem 1.3. The most natural choice for X would be the moduli stack of Galois representations, constructed in [EG23] when $G = \mathrm{GL}_n$. Since the case of more general groups has yet to be written up (though this is likely to be addressed by work of Lin, see for example [Lin23]) we take, in the body of the text, X equal to the formal spectrum of a Galois deformation ring.

The methods of this paper do not appear to give any way to prove an equality in Theorem 1.3. This would come down to showing that $[X^\mu \otimes \mathbb{F}] \leq [X_0^\mu]$ is an equality which ultimately, is a question about producing crystalline lifts with Hodge type μ of torsion Breuil–Kisin modules whose relative position in Gr_G is contained in $M_{\mu,\mathbb{F}}$. On the other hand, at least when $G = \mathrm{GL}_n$, equality could be obtained by proving the full support of patched modules for e.g. at Hodge types of the form $(\lambda+\rho,\dots,\rho)$. We hope to do this in future work.

Additional remarks.

- We don't know if Theorem 1.1 remains true without the bound on $\operatorname{char}\mathbb{F}$, or whether this bound, if necessary, is at all sharp. It is, however, worth observing that the bound in Theorem 1.1 is somewhat natural because of its relation to the irreducibility of the $W(\lambda)$ appearing in $\bigotimes_{i=1}^e W(\mu_i-\rho)$. More precisely, recall from [Jan03, 5.6] that, when $\mathbb{F}$ has positive characteristic, $W(\lambda)$ is simple when viewed as an algebraic representation over $\mathbb{F}$ whenever $0 \leq \langle \alpha^\vee, \lambda+\rho\rangle \leq p$. Since $W(\lambda)$ appears in $\bigotimes_{i=1}^e W(\mu_i-\rho)$ only if $\lambda \leq \sum_{i=1}^e(\mu_i-\rho)$ each such $W(\lambda)$ will be simple if
$$\sum_{i=1}^e \langle \alpha^\vee, \mu_i\rangle \leq \operatorname{char}\mathbb{F} + (e-1)\max_{\alpha^\vee}\langle \alpha^\vee, \rho\rangle$$
If $\max_{\alpha^\vee}\langle\alpha^\vee,\rho\rangle = 1$ (e.g. if $G = \mathrm{GL}_2$) then this is exactly the bound in Theorem 1.1 and so one might hypothesise that Theorem 1.1 remains true at least under this stronger bound. On the other hand, the irreducibility of the $W(\lambda)$ does not appear to play any direct role in our methods, making the significance of these observations questionable.
- In contrast, the stronger bound on the μ in Theorem 1.3 is far more unnatural. It arises from certain estimates in p-adic Hodge theory which could quite possibly be improved, at least so that they agree with the bound in Theorem 1.1.
- The requirement in Theorem 1.3 that the μ_κ be strictly dominant arises only from its appearance in Theorem 1.1. In particular, if a version of Theorem 1.1 could be proven for not necessarily strictly dominant cocharacters then the same arguments would allow any such cycle identities to be transferred to an inequality of cycles of crystalline representations with irregular Hodge types.
- While we suppress it from the notation we are not able to show that the $M_{\mu,\mathbb{F}}$ do not depend upon the choice of $\mathcal{O}$ and the $\pi_1,\dots,\pi_e$. Indeed, a more natural construction of the $M_{\mu,\mathbb{F}}$ would involve taking M_μ as the closure inside the Beilinson–Drinfeld grassmannian over $\mathbb{A}^e_{\mathbb{Z}}$ of an embedding an e-fold product of flag varieties over the locus of pairwise distinct tuples in $\mathbb{A}^e_{\mathbb{Z}}$. Then one could define $M_{\mu,\mathbb{F}}$ as the fibre over $0 \in \mathbb{A}^e_{\mathbb{F}}$, which would be independent of any choices. The problem is that, with this definition, we would need to know that $M_{\mu,\mathbb{F}}$ is flat around $0 \in \mathbb{A}^e_{\mathbb{Z}}$ and this is probably a rather subtle question (the analogous assertion for Schubert varieties is unknown for G not of type A).

Connections to previous work. We conclude by saying a little about how this paper relates to previous work. Concrete results so far towards Breuil–Mézard fall into two broad categories and (with a few exceptions that we mention shortly) all consider the case of GL_n. The first category considers the situation where $G = \mathrm{GL}_2$ and $K = \mathbb{Q}_p$. Here the conjecture is now essentially known, see [Kis09a, Paš15, HT15, San14, Tun21]. These results rely on the existence of a form of the p-adic Langlands correspondence, and therefore have little direct relation to our work.

The second category treats the conjecture in either higher dimensions and/or with K a finite extension of $\mathbb{Q}_p$, but at the cost of making (as we do) very strong assumptions on the size of the Hodge types appearing. For example, [GK14] proves the conjecture for any $K/\mathbb{Q}_p$ and two dimensional potentially crystalline representation of G_K with Hodge type $(0,1)$, while [LLHLM18] proves the conjecture for $K/\mathbb{Q}_p$ unramified and n-dimensional (tamely) potentially crystalline representation of G_K whose Hodge type is bounded by an inexplicit formula in terms of p and the tame type (which, at the very least, requires the Hodge types to be $\leq p$)

While the assumptions of both [GK14] and [LLHLM18] are entirely perpendicular to ours (in situations where the assumptions overlap the statement of Breuil–Mézard is vacuous) their methods are much closer in sprit to those of our paper. Indeed, both use moduli spaces of Breuil–Kisin modules to control moduli spaces of Galois representations, and describe the former moduli spaces in terms of closed subschemes inside an affine grassmannian. This is particularly true of the closed subschemes appearing in [LLHLM18] which are defined as a degeneration of a single flag variety (recall they consider $e = 1$) in an affine flag variety (i.e. a twisting of Gr_G which accounts for the tame type). Clearly, combining this definition with our construction of $M_{\mu,\mathbb{F}}$ describes candidate closed subschemes modelling the geometry of Breuil–Kisin modules associated to potentially crystalline representations of any finite extension of $\mathbb{Q}_p$ (at least for small Hodge types). On the other hand, there are significant points of departure from our methods and those of [GK14] and [LLHLM18]. While we use the control of moduli of Breuil–Kisin modules to directly analyse the special fibres of moduli of Galois representations, in loc. cit. they are used as a means to prove modularity lifting theorems, which are in turn used to control the moduli spaces of local Galois representations using patched modules.

Finally, while the majority of work towards Breuil–Mézard has focused on the case of GL_n, there has also been considerations of other groups. In [GG15] and [Dot18] the conjecture is considered for the group of units in a central division algebra, and in the latter it is shown that these conjectures follow from the conjecture for GL_n. In [DR22] the conjecture for PGL_n is also shown to follow from the case of GL_n. We also mention [Lev15] and [BL20] which use methods similar to ours to describe some deformation rings on crystalline representations valued in split reductive G.

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Part 1. Cycles identities in the affine grassmannian

2. Notation

2.1. Let $\mathcal{O}$ be a discrete valuation ring with residue field $\mathbb{F}$ and fraction field E. Let G be a split reductive group over $\operatorname{Spec}\mathcal{O}$ with connected fibres together with a choice of maximal torus T. Let $X_*(T) = \operatorname{Hom}(\mathbb{G}_m, T)$ and $X^*(T) = \operatorname{Hom}(T, \mathbb{G}_m)$ and write $\langle\ ,\ \rangle$ for the natural pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$. Let $R^\vee \subset X^*(T)$ be the roots (G,T) and for $\alpha^\vee \in R^\vee$ write $\alpha \in X_*(T)$ for the corresponding coroot. Let W be the Weyl group of (G,T). Choose a set of positive roots $R^\vee_+ \subset R^\vee$, with associated Borel B. Let R_+ denote the corresponding set of positive coroots. Recall

$$\lambda^\vee \leq \mu^\vee \Leftrightarrow \mu^\vee - \lambda^\vee \in \sum_{\alpha\in R^\vee_+} \mathbb{Z}_{\geq 0}\alpha^\vee$$

and that $\lambda^\vee \in X^*(T)$ is dominant if $\langle \lambda^\vee, \alpha\rangle \geq 0$ for all positive coroots $\alpha \in R_+$. We say $\lambda^\vee$ is strictly dominant if $\langle \lambda^\vee, \alpha\rangle \geq 1$. We likewise make sense of $\leq$ on $X_*(T)$ as well as dominant and strictly dominant $\lambda \in X_*(T)$.

Definition 2.2. An element $\rho \in X_*(T)$ is called a twisting element if $\langle \alpha^\vee, \rho\rangle = 1$ for all simple roots $\alpha^\vee$. Similarly $\rho^\vee \in X^*(T)$ is a twisting element if $\langle \rho^\vee, \alpha\rangle = 1$ for all simple coroots α.

Notice that if $\rho \in X_*(T)$ is a twisting element then $\rho - \frac{1}{2}\sum_{\alpha\in R_+}\alpha$ is W-invariant. Also $\lambda \in X_*(T)$ is strictly dominant if and only if $\lambda - \rho$ is dominant.

3. Torsors

3.1. Throughout this paper we view G-torsors from the following two equivalent viewpoints:

- A G-torsor $\mathcal{E}$ on $\operatorname{Spec} A$ is an A-scheme equipped with an action of G so that fppf (equivalently etale) locally on A one has $\mathcal{E} \cong G \times_{\mathcal{O}} \operatorname{Spec} A$.
- A fibre functor (a faithful exact tensor functor) from the category of representations of G on finite free $\mathcal{O}$-modules into the category of projective A-modules.

Any G-torsor in the first sense induces a fibre functor which sends a representation $\chi : G \to \mathrm{GL}(V)$ onto the contracted product

$$\mathcal{E}^\chi := \mathcal{E} \times^\chi V = \mathcal{E} \times V/\sim$$

That this construction produces an equivalence of categories is proved in e.g. [Bro13, 4.8]. We always write $\mathcal{E}^0$ for the trivial G-torsor and a trivialisation of a G-torsor on $\operatorname{Spec} A$ is an isomorphism $\mathcal{E} \cong \mathcal{E}^0$ over $\operatorname{Spec} A$.

4. Affine grassmannians

Fix an integer $e \geq 1$ and pairwise distinct $\pi_1, \ldots, \pi_e$ in the maximal ideal of $\mathcal{O}$. For any $\mathcal{O}$-algebra A we write $E(u) = \prod_{i=1}^e (u - \pi_i) \in A[u]$.

Definition 4.1. Let Gr_G denote the projective ind-scheme over $\mathcal{O}$ whose A-points, for any $\mathcal{O}$-algebra A, classify isomorphism classes of pairs $(\mathcal{E}, \iota)$ where

- $\mathcal{E}$ is a G-torsor over $\operatorname{Spec} A[u]$,

- ι is a trivialisation of $\mathcal{E}$ over the open subscheme $\operatorname{Spec} A[u, E(u)^{-1}]$, i.e. an isomorphism
$$\mathcal{E}|_{\operatorname{Spec} A[u,E(u)^{-1}]} \cong \mathcal{E}^0|_{\operatorname{Spec} A[u,E(u)^{-1}]}$$
where $\mathcal{E}^0$ denotes the trivial G-torsor.

We also consider variants $\mathrm{Gr}_{G,i}$ of Gr_G for $i = 1, \dots, e$ which are again projective ind-schemes over $\mathcal{O}$, and whose A-points classify isomorphism classes of pairs $(\mathcal{E}, \iota)$ with $\mathcal{E}$ a G-torsor on $\operatorname{Spec} A[u]$ and ι is a trivialisation of $\mathcal{E}$ over the open subscheme $\operatorname{Spec} A[u, (u-\pi_i)^{-1}]$. Notice that for each i there are natural closed immersions $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_G$. Each of Gr_G and $\mathrm{Gr}_{G,i}$ are also functorial in G.

Remark 4.2. When $G = \mathrm{GL}_n$ the above functor is a colimit over $a \geq 0$ of the functors sending an $\mathcal{O}$-algebra A onto the set of rank n projective $A[u]$ submodules
$$E(u)^a A[u]^n \subset \mathcal{E} \subset E(u)^{-a} A[u]^n$$
Since a submodule $\mathcal{E} \subset E(u)^a A[u]^n$ is $A[u]$-projective of rank n if and only if $E(u)^a A[u]^n/\mathcal{E}$ is A-projective (see [Zhu17, Lemma 1.1.5]) each subfunctor is represented by a subfunctor of the grassmannian classifying projective A-submodules of $E(u)^{-a} A[u]^n / E(u)^a A[u]^n$, which shows the ind-representability of $\mathrm{Gr}_{\mathrm{GL}_n}$.

For general G one chooses a faithful representation into GL_n and, using [Zhu17, 1.2.6], identifies Gr_G as a closed sub-indscheme of $\mathrm{Gr}_{\mathrm{GL}_n}$.

Lemma 4.3. *For any $\mathcal{O}$-algebra A set $\widehat{A[u]}_{E(u)}$ equal the $E(u)$-adic completion of $A[u]$. Then the A-valued points of Gr_G functorially identify with isomorphism classes of G-torsors on $\operatorname{Spec} \widehat{A[u]}_{E(u)}$ together with a trivialisation after inverting $E(u)$. Similarly for A-valued points of $\mathrm{Gr}_{G,i}$ with $E(u)$ replaced by $(u - \pi_i)$.*

Proof. This follows from the Beauville–Laszlo gluing lemma [BL95]. □

4.4. If A is an E-algebra and $n_i \in \mathbb{Z}_{\geq 0}$ then, since $E[u]$ is principal ideal domain, the product of the quotient maps describes an isomorphism
$$\frac{A[u]}{\prod_{i=1}^e (u-\pi_i)^{n_i}} \cong \prod_{i=1}^e \frac{A[u]}{(u-\pi_i)^{n_i}}$$
In particular, this gives an isomorphism $\widehat{A[u]}_{E(u)} \cong \prod_{i=1}^e \widehat{A[u]}_{(u-\pi_i)}$ where the completions are respectively taken against the ideals generated by $E(u)$ and $(u - \pi_i)$. As a consequence, we obtain:

Corollary 4.5. *There is an isomorphism*
$$\mathrm{Gr}_G \otimes_{\mathcal{O}} E \cong (\mathrm{Gr}_{G,1} \times_{\mathcal{O}} \ldots \times_{\mathcal{O}} \mathrm{Gr}_{G,e}) \otimes_{\mathcal{O}} E$$
written $(\mathcal{E}, \iota) \mapsto (\mathcal{E}_i, \iota_i)_{i=1,\dots,e}$ on A-valued points, so that
$$\mathcal{E} \otimes_{A[u]} \widehat{A[u]}_{E(u)} = \prod_{i=1}^e \mathcal{E}_i \otimes_{A[u]} \widehat{A[u]}_{u-\pi_i}$$
with $\iota = \prod_i \iota_i$.

4.6. The isomorphism in Corollary 4.5 has an alternative description. For any $\mathcal{O}$-algebra A consider the open subsets
$$U_i = \operatorname{Spec} A[u, \prod_{j \neq i} (u - \pi_j)^{-1}], \qquad V_i = \operatorname{Spec} A[u, (u-\pi_i)^{-1}]$$

of $\operatorname{Spec} A[u]$. Then $V_i = \bigcup_{j\neq i} U_j$, $V_i \cap U_i = \operatorname{Spec} A[u, E(u)^{-1}]$ and, if A is an E-algebra, then

$$\operatorname{Spec} A[u] = \bigcup_i U_i = V_i \cup U_i$$

Then $\mathcal{E}_i$ is the G-torsor obtained by glueing $\mathcal{E}|_{U_i}$ and $\mathcal{E}^0|_{V_i}$ along the restriction of ι to $U_i \cap V_i = \operatorname{Spec} A[u, E(u)^{-1}]$

Notation 4.7. For $\lambda \in X_*(T)$ write $\mathcal{E}_{\lambda,i}$ for the $\mathcal{O}$-valued point

$$(\mathcal{E}^0, (u - \pi_i)^\lambda) \in \mathrm{Gr}_{G,i}$$

where $\mathcal{E}^0$ denotes the trivial G-torsor on $\operatorname{Spec}\mathcal{O}[u]$ and $(u-\pi_i)^\lambda$ denotes the automorphism of $\mathcal{E}^0|_{\operatorname{Spec}\mathcal{O}[u,E(u)^{-1}]}$ induced by multiplication by $\lambda(u-\pi_i) \in G(\mathcal{O}[u, E(u)^{-1}])$. We then define the locally closed subscheme

$$\mathrm{Gr}_{G,i,\lambda} \subset \mathrm{Gr}_{G,i}$$

as the orbit of $\mathcal{E}_{\lambda,i}$ under the action of the group scheme L^+G with A-valued points $G(A[u])$.

Lemma 4.8. *For each $\lambda \in X_*(T)$ and $i = 1, \dots, e$ the morphism $G \to \mathrm{Gr}_{G,i}$ given by $g \mapsto g\mathcal{E}_{\lambda,i}$ induces a closed immersion*

$$G/P_\lambda \to \mathrm{Gr}_{G,i}$$

where

$$P_\lambda = \{g \in G \mid \lim_{t\to 0} \lambda(t)^{-1} g \lambda(t) \text{ exists}\}$$

Equivalently, P_λ is the parabolic subgroup of G generated by T and the roots subgroups $U_{\alpha^\vee}$ for $\alpha^\vee \in R^\vee$ with $\langle \alpha^\vee, \lambda\rangle \leq 0$.

Proof. The A-points of the stabiliser in G of $\mathcal{E}_\lambda$ consists of those $g \in G(A)$ for which $\lambda(u-\pi_i)^{-1} g \lambda(u - 1_i) \in G(A[u])$. Therefore, this stabiliser is precisely P_λ and we obtain a monomorphism $G/P_\lambda \to \mathrm{Gr}_{G,i}$. Since P_λ is a parabolic subgroup of G the quotient G/P_λ is proper over $\mathcal{O}$. Thus $G/P_\lambda \to \mathrm{Gr}_{G,i}$ is also proper. Proper monomorphisms are closed immersions [Sta17, 04XV] so the lemma follows. $\square$

Definition 4.9. For $\mu = (\mu_1, \dots, \mu_e)$ with $\mu_i \in X_*(T)$ set $M_\mu \subset \mathrm{Gr}_G$ equal to the scheme theoretic image of the composite

$$(G/P_{\mu_1} \times_{\mathcal{O}} \dots \times_{\mathcal{O}} G/P_{\mu_e}) \otimes_{\mathcal{O}} E \to (\mathrm{Gr}_{G,1} \times_{\mathcal{O}} \dots \times_{\mathcal{O}} \mathrm{Gr}_{G,e}) \otimes_{\mathcal{O}} E \cong \mathrm{Gr}_G \otimes_{\mathcal{O}} E \to \mathrm{Gr}_G$$

(the isomorphism coming from Lemma 4.5).

4.10. The A-valued points of G/P_λ classify filtrations of type λ on the trivial G-torsor over $\operatorname{Spec} A$ (i.e exact tensor functors from the category of representations of G on finite free $\mathcal{O}$-modules V into the category of filtrations on V by A-modules with projective graded pieces). From this point of view:

- The closed immersion $G/P_\lambda \to \mathrm{Gr}_{G,i}$ from Lemma 4.8 sends a filtration $\mathrm{Fil}^\bullet$ onto

$$\mathrm{Fil}^0(G \otimes_A \widehat{A[u]}_{(u-\pi)})$$

  where the filtration on $G \otimes_A \widehat{A[u]}_{(u-\pi)}$ is the tensor product of of $\mathrm{Fil}^\bullet$ with the $(u-\pi_i)$-adic filtration on $\widehat{A[u]}_{(u-\pi)}$.

- The closed immersion $(G/P_{\mu_1} \times_{\mathcal{O}} \ldots \times_{\mathcal{O}} G/P_{\mu_e}) \to \mathrm{Gr}_G$ from Definition 4.9 sends an A-valued point corresponding to an e-tuple of filtrations $(\mathrm{Fil}_i^\bullet)_i$ onto
$$\mathrm{Fil}^0(G \otimes_A \widehat{A[u]}_{E(u)})$$
where the filtration on $G \otimes_A \widehat{A[u]}_{E(u)} \cong \prod_{i=1}^e G \otimes_A \widehat{A[u]}_{u-\pi_i}$ is the i-fold product of the tensor product of $\mathrm{Fil}_i^\bullet$ and the $(u-\pi_i)$-adic filtration on $\widehat{A[u]}_{(u-\pi)}$.

5. Various Schubert varieties

Here we introduce some variants on the usual notation of Schubert varieties inside the affine grassmannian.

Notation 5.1. For $\lambda \in X_*(T)$ write $\mathcal{E}_{\lambda,\mathbb{F}} \in \mathrm{Gr}_G(\mathbb{F})$ for the point corresponding to $(\mathcal{E}^0, u^\lambda)$. Thus
$$\mathcal{E}_{\lambda,\mathbb{F}} = \mathcal{E}_{\lambda,i} \otimes_{\mathcal{O}} \mathbb{F}$$
with $\mathcal{E}_{\lambda,i}$ as defined in Notation 4.7 and for any $i = 1,\ldots,e$. We also write $\mathrm{Gr}_{G,\lambda,\mathbb{F}} = \mathrm{Gr}_{G,\lambda,i} \otimes_{\mathcal{O}} \mathbb{F}$ for any $i = 1,\ldots,e$. Equivalently, $\mathrm{Gr}_{G,\lambda,\mathbb{F}}$ is the L^+G-orbit of $\mathcal{E}_{\lambda,\mathbb{F}}$.

For $\lambda \in X_*(T)$ the Schubert variety $\mathrm{Gr}_{G,\leq\lambda,\mathbb{F}} \subset \mathrm{Gr}_G \otimes_{\mathcal{O}} \mathbb{F}$ is usually defined as the closure of $\mathrm{Gr}_{G,\lambda,i} \otimes_{\mathcal{O}} \mathbb{F}$ (for any $i = 1,\ldots,e$). Then $\mathrm{Gr}_{G,\leq\lambda,\mathbb{F}}$ is reduced, irreducible, and can be expressed as
$$\mathrm{Gr}_{G,\leq\lambda,\mathbb{F}} = \bigcup_{\lambda' \leq \lambda} \mathrm{Gr}_{G,\lambda',\mathbb{F}}$$
We would like to use $\mathrm{Gr}_{G,\leq\mu_1+\ldots+\mu_e,\mathbb{F}}$ as an ambient space in which to study $M_\mu \otimes_{\mathcal{O}} \mathbb{F}$. However, the containment of $M_\mu \otimes_{\mathcal{O}} \mathbb{F}$ in $\mathrm{Gr}_{G,\leq\mu_1+\ldots+\mu_e,\mathbb{F}}$ is unclear due to both varieties construction as a closure. For this reason we instead use a moduli construction which is close to (and conjectured to equal) $\mathrm{Gr}_{G,\leq\lambda,\mathbb{F}}$.

Definition 5.2. Let V be a free $\mathcal{O}$-module. Any A-valued point of Gr_G corresponds to a projective $A[u]$-submodule of $V \otimes_{\mathcal{O}} A[u, E(u)^{-1}]$ and so, for any e-tuple of integers (n_i), we may consider the closed subfunctor $Y^{\geq(n_i)}_{\mathrm{GL}(V)} \subset \mathrm{Gr}_{\mathrm{GL}(V)}$ consisting of those $\mathcal{E}$ for which
$$\mathcal{E} \subset \prod_{i=1}^e (u-\pi_i)^{n_i} V \otimes_{\mathcal{O}} A[u] \tag{5.3}$$
If $\mu = (\mu_1,\ldots,\mu_e)$ with $\mu_i \in X_*(T)$ dominant then define
$$Y_{G,\leq\mu} = \bigcap_\chi \left(\mathrm{Gr}_G \times_{\chi,\mathrm{Gr}_{\mathrm{GL}(V)}} Y^{\geq(\langle w_0\chi^\vee,\mu_i\rangle)}_{\mathrm{GL}(V)} \right)$$
where $w_0 \in W$ is the longest element and the intersection runs over irreducible algebraic representations $\chi : G \to \mathrm{GL}(V)$ of highest weight $\chi^\vee$.

Example 5.4. Suppose that $G = \mathrm{GL}_n$ so that A-valued points of Gr_G identify with projective $A[u]$-submodules $\mathcal{E} \subset A[u, E(u)^{-1}]^n$. If
$$\mu = (\mu_1,\ldots,\mu_e), \qquad \mu_i = (\mu_{i,1} \geq \ldots \geq \mu_{i,n})$$
then $\mathcal{E} \in Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$ then
$$\bigwedge^j(\mathcal{E}) \subset \prod_{i=1}^e (u-\pi_i)^{\mu_{i,n}+\ldots+\mu_{i,n-j+1}} A[u]^{\binom{n}{j}} \tag{5.5}$$

for all $j = 1, \dots, n$. Indeed, if $\chi : G \to \mathrm{GL}(V)$ equals the j-th exterior power of the standard representation, then the induced morphism $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}(V)}$ sends $\mathcal{E}$ onto $\bigwedge^j(\mathcal{E})$ and so

$$\mathrm{Gr}_G \times_{\chi, \mathrm{Gr}_{\mathrm{GL}(V)}} Y_{\mathrm{GL}(V)}^{\geq (\langle w_0 \chi^\vee, \mu_i \rangle)}$$

is the closed subscheme consisting of $\mathcal{E}$ as in (5.5). This is because

$$\chi^\vee = (\underbrace{1, \dots, 1}_{j \text{ ones}}, 0, \dots, 0)$$

and so $\langle w_0 \chi^\vee, \mu_i \rangle = \mu_{i,n} + \ldots + \mu_{i,n-j+1}$. In fact, since every highest weight representation of GL_n is a quotient of tensor products of these exterior powers representations, conditions (5.5) suffice to determine $Y_{G, \leq \mu}$.

The following lemma describes the basic properties of $Y_{G,\leq \mu}$ that we need.

Lemma 5.6. *Let $\mu_1, \dots, \mu_e \in X_*(T)$ be dominant. Then $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$ only depends upon $\mu_1 + \ldots + \mu_e$, contains $\mathrm{Gr}_{G, \mu_1 + \ldots + \mu_e, \mathbb{F}}$ as an open subset, and*

$$(Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F})_{\mathrm{red}} = \bigcup_{\lambda \leq \mu_1 + \ldots + \mu_e} \mathrm{Gr}_{G,\lambda,\mathbb{F}}$$

Proof. That $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$ only depends upon $\mu_1 + \ldots + \mu_e$ is clear from the definition. It is also clear from the definition that $Y_{G,\leq\mu}$ contains $\mathcal{E}_{\lambda,\mathbb{F}} \in \mathrm{Gr}_G(\mathbb{F})$ if and only if $\lambda \leq \mu_1 + \ldots + \mu_e$. This implies

$$(Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F})_{\mathrm{red}} = \bigcup_{\lambda \leq \mu_1 + \ldots + \mu_e} \mathrm{Gr}_{G,\lambda,\mathbb{F}}$$

It remains to show $\mathrm{Gr}_{G,\mu_1+\ldots,\mu_e,\mathbb{F}}$ is open in $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$. This will follow if $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$ is reduced at $\mathcal{E}_{\mu_1+\ldots+\mu_e,\mathbb{F}}$, which can be achieved by a simple tangent space computation identifying the tangent space of $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$ at $\mathcal{E}_{\mu_1+\ldots+\mu_e,\mathbb{F}}$ with the tangent space of $\mathrm{Gr}_{G,\mu_1+\ldots+\mu_e,\mathbb{F}}$. See [KMW18, §3]. □

Remark 5.7. The construction of $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$ was proposed in [FM99] as a moduli interpretation of $\mathrm{Gr}_{G,\leq\mu_1+\ldots+\mu_e,\mathbb{F}}$. However, it is an open question whether $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F} = \mathrm{Gr}_{G,\leq\mu_1+\ldots+\mu_e,\mathbb{F}}$ (equivalently, whether $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$ is reduced). The equality is known when $G = \mathrm{SL}_n$ and $\mathbb{F}$ has characteristic zero, see [KMW18].

Lemma 5.8. *Under the isomorphism $\mathrm{Gr}_G \otimes_{\mathcal{O}} E \cong (\mathrm{Gr}_{G,1} \times_{\mathcal{O}} \ldots \times_{\mathcal{O}} \mathrm{Gr}_{G,e}) \otimes_{\mathcal{O}} E$ from Lemma 4.5 we have*

$$Y_{G,\leq\mu} \otimes_{\mathcal{O}} E \cong (Y_{G,1,\leq\mu_1} \times_{\mathcal{O}} \ldots \times_{\mathcal{O}} Y_{G,e,\leq\mu_e}) \otimes_{\mathcal{O}} E$$

where $Y_{G,i,\leq\mu_i} \subset \mathrm{Gr}_{G,i}$ is defined as a special case of Definition 5.2.

Proof. The lemma reduces to showing that, for any tuple (n_i) of integers and any finite free $\mathcal{O}$-module V,

$$Y_{\mathrm{GL}(V)}^{\geq (n_i)} \otimes_{\mathcal{O}} E = \left(Y_{\mathrm{GL}(V),1}^{\geq n_1} \times_{\mathcal{O}} \ldots \times_{\mathcal{O}} Y_{\mathrm{GL}(V),e}^{\geq n_e} \right) \otimes_{\mathcal{O}} E$$

under the identification from Lemma 4.5. This is clear since if A is an E-algebra then the isomorphism $\widehat{A[u]}_{E(u)} \cong \prod_{i=1}^e \widehat{A[u]}_{u-\pi_i}$ identifies the ideal generated by $\prod (u-\pi_i)^{n_i}$ with the product of the ideals generated by $(u-\pi_i)^{n_i}$. □

The reason we introduce Definition 5.2 is because it easily allows us to prove:

Proposition 5.9. *For $\mu = (\mu_1, \dots, \mu_e)$ with $\mu_i \in X_*(T)$ dominant we have $M_\mu \subset Y_{G,\leq\mu}$.*

Proof. It suffices to show $M_\mu \otimes_{\mathcal{O}} E \subset Y_{G,\leq\mu}$ because $Y_{G,\leq\mu}$ is closed in Gr_G. By Lemma 5.8 we are reduced to showing that $G/P_{\mu_i} \to \mathrm{Gr}_{G,i}$ factors through $Y_{G,i,\leq\mu_i}$. Since $Y_{G,i,\leq\mu_i}$ is G-stable it is enough to show $\mathcal{E}_{\mu_i,i} \in Y_{G,i,\leq\mu_i}$, and this is clear. □

6. Approximations via Gr_G^∇

When $G = \mathrm{GL}_n$ the spaces $M_\mu \otimes_{\mathcal{O}} \mathbb{F}$ were accessed in [Bar21, §7] by constructing a subfunctor $\mathrm{Gr}_G^\nabla \subset \mathrm{Gr}_G$ with $M_\mu \subset \mathrm{Gr}_G^\nabla$. The subfunctor Gr_G^∇ consists of $\mathcal{E} \in \mathrm{Gr}_G(A)$ which, when viewed as projective $A[u]$-modules of $A((u))^n$, satisfy

$$E(u)\nabla(\mathcal{E}) \subset \mathcal{E}$$

for ∇ the operator on $A((u))^n$ given coordinate-wise via $\frac{d}{du}$. If $\mathcal{E}$ is generated by $(e_1, \ldots, e_n)X$ for a matrix $X \in \mathrm{GL}_n(A((u)))$ then this is equivalent to asking that

$$E(u)X^{-1}\frac{d}{du}(X) \in \mathrm{Mat}(A[[u]])$$

In this section we show how to extend this construction to general G.

6.1. The following construction works when $G = \mathrm{Spec}\,\mathcal{O}_G$ is any affine algebraic group over $\mathcal{O}$. Set $\mathfrak{g} = \mathrm{Lie}(G)$. In what follows we will interpret elements of $\mathfrak{g}$ as derivations $\mathcal{O}_G \to \mathcal{O}$ over $\mathcal{O}$ where $\mathcal{O}_G$ acts on $\mathcal{O}$ via the counit map $e : \mathcal{O}_G \to \mathcal{O}$. The logarithmic derivative can then be described as a map

$$\mathrm{dlog} : G(B) \to \mathfrak{g} \otimes_{\mathbb{Z}} \Omega_{B/\mathcal{O}}$$

for any ring B. To define this map identify $\Omega_{G/\mathcal{O}} = \mathcal{L}(G)^\vee \otimes_{\mathcal{O}} \mathcal{O}_G$ where $\mathcal{L}(G)$ denotes the translation invariant derivations $\mathcal{O}_G \to \mathcal{O}_G$. Then

$$\mathfrak{g} \otimes_{\mathcal{O}} \Omega_{G/\mathcal{O}} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{L}(G), \mathfrak{g}) \otimes_{\mathcal{O}} \mathcal{O}_G$$

and the map $\mathcal{L}(G) \to \mathfrak{g}$ given by composition with the counit e defines a canonical global section. For any $g \in G(B)$ define $\mathrm{dlog}(g)$ as the image of this section under $\mathfrak{g} \otimes_{\mathcal{O}} g^*\Omega_{G/\mathcal{O}} \to \mathfrak{g} \otimes_{\mathcal{O}} \Omega_{B/\mathcal{O}}$. If A is any $\mathcal{O}$-algebra and $B = A[u, E(u)^{-1}]$ then we obtain an element

$$\mathrm{dlog}_u(g) \in \mathfrak{g} \otimes_{\mathcal{O}} A[u, E(u)^{-1}]$$

by evaluating $\mathrm{dlog}(g)$ at the derivation $\frac{d}{du} : A[u, E(u)^{-1}] \to A[u, E(u)^{-1}]$. This construction is functorial in G.

The following example motivates us calling this construction the logarithmic derivative.

Example 6.2. Let $G = \mathrm{GL}_n$ with coordinates T_{ij} and write $\iota : \mathcal{O}_G \to \mathcal{O}_G$ for the coinverse map. Write $\frac{d}{dT_{ij}}$ for the element of $\mathfrak{g}$ sending $T_{ij} \mapsto 1$ and zero on all other coordinates. We claim that the section

$$\sum_{ij} \frac{d}{dT_{ij}} \otimes \left(\sum_m \iota(T_{im})d(T_{mj})\right) \in \mathfrak{g} \otimes_{\mathcal{O}} \Omega_{G/\mathcal{O}} \tag{6.3}$$

coincides with the map $\mathcal{L}(G) \to \mathfrak{g}$ given by composition with the counit. If $\Delta : T_{ij} \mapsto \sum_l T_{il} \otimes T_{lj}$ is the comultiplication map then $\mathcal{L}(G) \to \mathfrak{g}$ has an inverse given by $d \mapsto (\mathrm{id} \otimes d) \circ \Delta$ (see for example [Mil17, 12.24]). Therefore we can check the claim by evaluating $\sum_m \iota(T_{im})d(T_{mj})$ at $(\mathrm{id} \otimes \frac{d}{dT_{lk}}) \circ \Delta$. Since

$$(\mathrm{id} \otimes \frac{d}{dT_{lk}}) \circ \Delta : T_{mj} \mapsto \begin{cases} T_{ml} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

this evaluation is equal to

$$\begin{cases} \sum_m \iota(T_{im})T_{ml} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Since composing $(\iota \otimes \mathrm{id}) \circ \Delta$ with multiplication $\mathcal{O}_G \otimes \mathcal{O}_G \to \mathcal{O}_G$ equals the counit e it follows that the above evaluation is 1 if $ij = lk$ and zero otherwise. This verifies our claim. If $g = (g_{ij}) \in G(B)$ has inverse $g^{-1} = (h_{ij})$ then the image of (6.3) in $\mathfrak{g} \otimes \Omega_{B/\mathcal{O}}$ is

$$\mathrm{dlog}(g) = \sum_{ij} \frac{d}{dT_{ij}} \otimes \left(\sum_m h_{im} d(g_{mj}) \right)$$

If $B = A[u, E(u)^{-1}]$ and we identify $\mathfrak{g} = \mathrm{Mat}_{n\times n}(\mathcal{O})$ via $\frac{d}{dT_{ij}}$ then evaluating $\mathrm{dlog}(g)$ at $\frac{d}{du}$ yields $\mathrm{dlog}_u(g) = g^{-1}\frac{d}{du}(g)$.

Remark 6.4. If G is a flat and finite type over $\mathcal{O}$ then dlog_u can alternatively be constructed using the Tannakian viewpoint. As explained in e.g. [Lev15], an element of $\mathfrak{g} \otimes_{\mathcal{O}} A[u]$ is equivalent to a collection of endomorphisms X_V for all representations $G \to \mathrm{GL}(V)$ of G on finite free $\mathcal{O}$-modules which are compatible with exact sequences and satisfy $X_{V_1 \otimes V_2} = X_{V_1} \otimes 1 + 1 \otimes X_{V_2}$. Example 6.2 shows that $\mathrm{dlog}_u(g)$ corresponds to the rule sending a representation ρ onto

$$\rho(g)^{-1}\frac{d}{du}(\rho(g)) \in \mathrm{End}(V)$$

(the compatibility of this rule with exact sequences and tensor products being an easy computation).

Example 6.5. If $G = \mathbb{G}_a$ with coordinate T then write $\frac{d}{dT}$ for the element of $\mathfrak{g}$ sending $T \mapsto 1$. In this case the section

$$\frac{d}{dT} \otimes d(T) \in \mathfrak{g} \otimes_{\mathcal{O}} \Omega_{G/\mathcal{O}}$$

coincides with the map $\mathcal{L}(G) \to \mathfrak{g}$ given by composition with the counit and so if $g \in \mathbb{G}_a(B)$ then

$$\mathrm{dlog}(g) = \frac{d}{dT} \otimes d(b)$$

If $B = A[u, E(u)^{-1}]$ and we identify $\mathfrak{g} = \mathcal{O}$ via $\frac{d}{dT}$ it follows that $\mathrm{dlog}_u(g) = \frac{d}{du}(g)$.

The next lemma contains what we will need to compute with $\mathrm{dlog}_u(-)$.

Lemma 6.6. *(1) For $g, h \in G(A[u, E(u)^{-1}])$ we have*

$$\mathrm{dlog}_u(gh) = \mathrm{Ad}(h^{-1})\,\mathrm{dlog}_u(g) + \mathrm{dlog}_u(h)$$

where Ad *denotes the adjoint action of G on $\mathfrak{g}$.*

(2) $\mathrm{dlog}_u(g) = 0$ *for $g \in G(A)$*

(3) $u\,\mathrm{dlog}_u(u^\lambda) \in \mathrm{Lie}(T)$ *for $\lambda \in X_*(T)$.*

Proof. By choosing a faithful representation of G all these identities reduce to the case of GL_n, where they are clear given the interpretation of dlog_u from Example 6.2. □

Definition 6.7. Define $\mathrm{Gr}^{\nabla}_G \subset \mathrm{Gr}_G$ to be the subfunctor consisting of those $(\mathcal{E}, \iota) \in \mathrm{Gr}_G(A)$ for which there exist an fppf cover $A \to A'$ trivialising $\mathcal{E}$ so that, if $\iota \times_{A[u]} A'[u]$ is given by left multiplication by $g \in G(A'[u, E(u)^{-1}])$, then

$$E(u)\,\mathrm{dlog}_u(g) \in \mathfrak{g} \otimes_{\mathcal{O}} A[u]$$

This is a closed subfunctor since $A[u, E(u)^{-1}]/A[u]$ is a projective A-module. When $G = \mathrm{GL}_n$ this coincides with the subfunctor defined in [Bar21, 7.4]. Lemma 6.6 shows that Gr^{∇}_G is also G-stable.

Proposition 6.8. *$M_\mu \subset \mathrm{Gr}^{\nabla}_G$ for any $\mu = (\mu_1, \ldots, \mu_e)$.*

Proof. Since Gr^{∇}_G is a closed subfunctor it suffices to show that $M_\mu \otimes_{\mathcal{O}} E \subset \mathrm{Gr}^{\nabla}_G$. For this observe that, under the identification $\mathrm{Gr}_G \otimes_{\mathcal{O}} E \cong (\mathrm{Gr}_{G,1} \times_{\mathcal{O}} \ldots \times_{\mathcal{O}} \mathrm{Gr}_{G,e}) \otimes_{\mathcal{O}} E$, one has

$$\mathrm{Gr}^{\nabla}_G \otimes_{\mathcal{O}} E = \left(\mathrm{Gr}^{\nabla}_{G,1} \times_{\mathcal{O}} \ldots \times_{\mathcal{O}} \mathrm{Gr}^{\nabla}_{G,e}\right) \otimes_{\mathcal{O}} E$$

where $\mathrm{Gr}^{\nabla}_{G,i}$ is defined analogously to Gr^{∇}_G with the condition $E(u)\,\mathrm{dlog}_u(g) \in \mathfrak{g} \otimes_{\mathcal{O}} A[u]$ replaced by $(u - \pi_i)\,\mathrm{dlog}_u(g) \in \mathfrak{g} \otimes_{\mathcal{O}} A[u]$. We are therefore reduced to showing that the closed immersions $G/P_\lambda \to \mathrm{Gr}_{G,i}$ induced by any $\lambda \in X_*(T)$ factor through $\mathrm{Gr}^{\nabla}_{G,i}$, and this follows from Lemma 6.6. $\square$

7. Computations in Gr^{∇}_G

In this section we show that the inclusion $M_\mu \subset \mathrm{Gr}^{\nabla}_G$ induces a reasonable topological description of $M_\mu \otimes \mathbb{F}$ provided μ is sufficiently small relative to the characteristic of $\mathbb{F}$. As with the previous section, this extend results from [Bar21, §7] beyond $G = \mathrm{GL}_n$.

Proposition 7.1. *Suppose that $\lambda \in X_*(T)$ is dominant. If $\mathrm{char}\,\mathbb{F} > 0$ then assume that*

$$\langle \alpha^\vee, \lambda \rangle \leq \mathrm{char}\,\mathbb{F} + e - 1$$

for every positive root $\alpha^\vee$. Then

$$\mathrm{Gr}_{G,\lambda,\mathbb{F}} \times_{\mathrm{Gr}_G} \mathrm{Gr}^{\nabla}_G$$

is smooth and irreducible of dimension

$$\sum_{\alpha \in R^+} \min\{e, \langle \alpha, \lambda \rangle\}$$

Proof. As in the previous section we write $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathfrak{t} = \mathrm{Lie}(T)$ and write

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha^\vee \in R^\vee} \mathfrak{g}_{\alpha^\vee}$$

for the root decomposition of $\mathfrak{g}$. For each $\alpha^\vee \in R^\vee$ we have the associated root homomorphism $x_{\alpha^\vee} : \mathbb{G}_a \to G$ which induces an identification $dx_{\alpha^\vee} : \mathbb{G}_a \xrightarrow{\sim} \mathfrak{g}_{\alpha^\vee}$. See for example [Jan03, 1.2].

Step 1. We begin by recalling a standard open cover of G/P_λ. Let U denote the image of the morphism

$$\prod_{\langle \alpha^\vee, \lambda \rangle > 0} \mathbb{A}^1 \to G$$

given by $(a_{\alpha^\vee})_{\alpha^\vee} \mapsto \prod_{\alpha^\vee} x_{\alpha^\vee}(a_{\alpha^\vee})$ (the product taken in an arbitrary, but fixed, order). Then U is a closed subgroup of G and the induced morphism $U \to G/P_\lambda$ is an open immersion. Furthermore, the W-translates of the image of U form an open cover of G/P_λ. See [Jan03, 1.10] for more details.

Step 2. Let U_λ denote the image of the morphism

$$\prod_{\langle\alpha^\vee,\lambda\rangle>0}\mathbb{A}^{\langle\alpha^\vee,\lambda\rangle}\to L^+G$$

given by $(a_{\alpha^\vee,0},a_{\alpha,1},\dots,a_{\alpha^\vee,\langle\alpha^\vee,\lambda\rangle-1})_{\alpha^\vee}\mapsto\prod_{\alpha^\vee}x_{\alpha^\vee}(\sum_i a_{\alpha^\vee,i}u^i)$ (again the product is taken in an arbitrary, but fixed, order). Then the morphism $U_\lambda\to\mathrm{Gr}_{G,\lambda,\mathbb{F}}$ given by $g\mapsto gu^\lambda$ is an open immersion whose W-translates cover $\mathrm{Gr}_{G,\lambda,\mathbb{F}}$. To see this note first that $U_\lambda\to\mathrm{Gr}_{G,\lambda,\mathbb{F}}$ is a monomorphism. Secondly, note that $U_\lambda\to\mathrm{Gr}_{G,\lambda,\mathbb{F}}$ factors through the preimage of U under the morphism $q:\mathrm{Gr}_{G,\lambda,\mathbb{F}}\to G/P_\lambda$ sending gu^λ onto g modulo u. Thirdly, note that $q^{-1}(U)$ is smooth and irreducible of dimension $\sum_{\alpha\in R^+}\langle\alpha^\vee,\lambda\rangle=\langle 2\rho^\vee,\lambda\rangle$ (because the same is known to be true of $\mathrm{Gr}_{G,\lambda,\mathbb{F}}$, see for example [Zhu17, 2.1.5]). Therefore $U_\lambda\to q^{-1}(U)$, being a monomorphism between integral schemes of the same dimension, is an isomorphism (because monomorphisms are unramified [Sta17, 02GE] and unramified morphisms are etale locally closed immersions [Liu02, 4.11]).

Step 3. We are going to compute the closed subscheme $U_\lambda\times_{\mathrm{Gr}_G}\mathrm{Gr}_G^\nabla$. By definition $g\in U_\lambda(A)$ is contained in this closed subscheme if and only if

$$u^e\,\mathrm{dlog}_u(gu^\lambda)\in\mathfrak{g}\otimes_{\mathcal{O}}A[u]$$

Lemma 6.6 shows this is equivalent to asking that

$$u^e\,\mathrm{Ad}(u^{-\lambda})\,\mathrm{dlog}_u(g)\in\mathfrak{g}\otimes_{\mathcal{O}}A[u]$$

It will therefore be necessary to compute $\mathrm{dlog}_u(g)$ and we will do this using the following two observations:

- If $g=x_{\alpha^\vee}(a)$ for $a\in A[u]$ then $\mathrm{dlog}_u(g)=dx_{\alpha^\vee}(\frac{d}{du}a)$. This follows from Example 6.5 and the functoriality of dlog_u.
- If $\alpha^\vee+\beta^\vee\neq 0$ then

$$\mathrm{Ad}(x_{\alpha^\vee}(a))dx_{\beta^\vee}(b)=dx_{\beta^\vee}(b)+\sum_{i,j>0}c_{ij}d_{i\alpha^\vee+j\beta^\vee}(a^ib^j)$$

  for some $c_{ij}\in\mathbb{Z}$ independent of a and b. This can be seen by passing the formula

$$x_{\alpha^\vee}(a)x_{\beta^\vee}(b)x_{\alpha^\vee}(a)^{-1}=x_{\beta^\vee}(b)\prod_{i,j>0}x_{i\alpha^\vee+j\beta^\vee}(c_{ij}a^ib_j)$$

  found in e.g. [Jan03, 1.2.(5)] to the Lie algebra.

Step 4. Lemma 6.6 shows that $\mathrm{dlog}_u(gx_{\beta^\vee}(b))=\mathrm{Ad}(x_{\beta^\vee}(-b))\,\mathrm{dlog}_u(g)+\mathrm{dlog}_u(x_{\beta^\vee}(b))$. This, together with the two bullet points from Step 3, allows an inductive computation of $\mathrm{dlog}_u(g)$ for $g=\prod_{\langle\alpha^\vee,\lambda\rangle>0}x_{\alpha^\vee}(a_{\alpha^\vee})$ with $a_{\alpha^\vee}\in A[u]$. We see that $\mathrm{dlog}_u(g)$ can be expressed as a sum, over $\gamma^\vee$ with $\langle\gamma^\vee,\lambda\rangle>0$, of terms

$$dx_{\gamma^\vee}\left(\frac{d}{du}a_{\gamma^\vee}+C_{\gamma^\vee}\right)$$

where $C_{\gamma^\vee}$ is a $\mathbb{Z}$-linear combination of products of the $a_{\alpha^\vee}$ and $\frac{d}{du}(a_{\alpha^\vee})$ for those roots $\alpha^\vee$ with $\langle\alpha^\vee,\lambda\rangle>0$ and $\langle\gamma^\vee-\alpha^\vee,\lambda\rangle>0$. We can therefore write $C_{\gamma^\vee}=C_{\gamma^\vee,0}+C_{\gamma^\vee,1}u+C_{\gamma^\vee,2}u^2+\dots$ with each $C_{\gamma^\vee,i}=C_{\gamma^\vee,i}(a_{\alpha^\vee,j})$ a polynomial in the coefficients of the $a_{\alpha^\vee}$ for $\alpha^\vee$ with $0 < \langle \alpha^\vee, \lambda\rangle < \langle \gamma^\vee, \lambda\rangle$. These polynomials have $\mathbb{Z}$-coefficients and depend only on the order in which the product defining g is taken. It follows that $u^e \operatorname{Ad}(u^{-\lambda}) \operatorname{dlog}_u(g)$ can likewise be expressed as a sum of the terms

$$u^{e-\langle \gamma^\vee,\lambda\rangle} dx_{\gamma^\vee} \left(\frac{d}{du} a_{\gamma^\vee} + C_{\gamma^\vee} \right) \tag{7.2}$$

The assumption that $\langle \gamma^\vee, \lambda\rangle - e + 1 \leq \operatorname{char} \mathbb{F}$ means there exist unique polynomials $D_{\gamma^\vee,i} = D_{\gamma^\vee,i}(a_{\alpha^\vee,i})$ in the coefficients of the a_α for $\alpha^\vee$ with $0 < \langle \alpha^\vee, \lambda\rangle < \langle \gamma^\vee, \lambda\rangle$ so that if

$$D_{\gamma^\vee} = D_{\gamma^\vee,1} u + D_{\gamma^\vee,2} u^2 + \ldots + D_{\gamma^\vee, \langle\gamma^\vee,\lambda\rangle - e+1} u^{\langle \gamma^\vee,\lambda\rangle - e+1}$$

then $\frac{d}{du} D_{\gamma^\vee} \equiv C_{\gamma^\vee}$ modulo $u^{\langle \gamma^\vee,\lambda\rangle - e+1}$. Again these polynomials depend only on the order of the product defining g. Thus (7.2) is contained in $\mathfrak{g}_{\gamma^\vee} \otimes_{\mathcal{O}} A[u]$ if and only if

$$a_{\gamma^\vee} - D_{\gamma^\vee} \in A + u^{\langle \gamma^\vee,\lambda\rangle - e+1} A[u]$$

It follows that there is an isomorphism

$$\prod_{\langle \gamma^\vee,\lambda\rangle > 0} \mathbb{A}^{\min\{e, \langle \gamma^\vee,\lambda\rangle\}} \to U_\lambda \times_{\operatorname{Gr}_G} \operatorname{Gr}_G^\nabla$$

sending $(a_{\gamma^\vee,i})_{\gamma^\vee}$ onto

$$\prod_{\gamma^\vee} x_{\gamma^\vee} \left(a_0 + a_{\gamma^\vee,1} u^{\langle \gamma^\vee,\lambda\rangle - 1} + a_{\gamma^\vee,2} u^{\langle \gamma^\vee,\lambda\rangle - 2} + \ldots + a_{\gamma^\vee, \min\{e,\langle\gamma^\vee,\lambda\rangle - \}-1} u^{\max\{1,\langle \gamma^\vee,\lambda\rangle\} - e+1} + D_{\gamma^\vee} \right)$$

This shows that $U_\lambda \times_{\operatorname{Gr}_G} \operatorname{Gr}_G^\nabla$ is smooth of the claimed dimension.

Step 5. It remains to show that $\operatorname{Gr}_{G,\lambda,\mathbb{F}} \times_{\operatorname{Gr}_G} \operatorname{Gr}_G^\nabla$ is irreducible. For this recall the action of $\mathbb{G}_m$ on Gr_G via loop rotations: if $t \in A^\times$ and $(\mathcal{E}, \iota) \in \operatorname{Gr}_{G,A}$ then

$$t \cdot (\mathcal{E}, \iota) = (x_t^* \mathcal{E}, x_t^* \iota)$$

where x_t is the automorphism of $\operatorname{Spec} A[u]$ given by $u \mapsto tu$. This action stabilises both $\operatorname{Gr}_{G,\lambda,\mathbb{F}}$ and $\operatorname{Gr}_G^\nabla$. Therefore, smoothness of $\operatorname{Gr}_{G,\lambda,\mathbb{F}} \times_{\operatorname{Gr}_G} \operatorname{Gr}_G^\nabla$ ensures it is an affine bundle over its $\mathbb{G}_m$-fixed points, see [Mil17, Theorem 13.47]. Since the fixed point locus in $\operatorname{Gr}_{G,\lambda,\mathbb{F}}$ is the G-orbit of $\mathcal{E}_{\lambda,\mathbb{F}}$, and since this is contained in $\operatorname{Gr}_G^\nabla$, we conclude that the fixed point locus of $\operatorname{Gr}_{G,\lambda,\mathbb{F}} \times_{\operatorname{Gr}_G} \operatorname{Gr}_G^\nabla$ is also this G-orbit. As this orbit is irreducible the same is true for $\operatorname{Gr}_{G,\lambda,\mathbb{F}} \times_{\operatorname{Gr}_G} \operatorname{Gr}_G^\nabla$. $\square$

8. Naive cycle identities

Here we use Proposition 7.1 to produce a basic description of the cycles associated to $M_\mu \otimes_{\mathcal{O}} \mathbb{F}$.

Definition 8.1. A d-dimensional cycle on any Noetherian (ind)-scheme X is a $\mathbb{Z}$-linear combination of integral closed subschemes in X of dimension d. The group of all such cycles is denoted $Z_d(X)$. If $\mathcal{F}$ is any coherent sheaf on X we write

$$[\mathcal{F}] = \sum_Z m(Z, \mathcal{F})[Z]$$

where the sum runs over d-dimensional integral closed subschemes Z in X and $m(Z,Y)$ denotes the $\mathcal{O}_{Z,\xi}$-dimension of $\mathcal{F}_\xi$ for $\xi \in Z$ the generic point. If $i : Y \subset X$ is a closed subscheme then we set $[Y] = [i_* \mathcal{O}_Y]$. If X is a scheme then [Sta17, 02S9] shows that $Z_d(X)$ can alternatively be defined as the cokernel of the map

$$K_0(\operatorname{Coh}_{\leq d-1}(X)) \to K_0(\operatorname{Coh}_{\leq d}(X))$$

where $\mathrm{Coh}_{\leq d}(X)$ denotes the category of coherent sheaves on X with support of dimension $\leq d$. Then $[\mathcal{F}]$ coincides with the image of the class of $\mathcal{F}$.

Definition 8.2. For $\lambda \in X_*(T)$ define $\mathcal{C}_\lambda \subset \mathrm{Gr}_G \otimes_{\mathcal{O}} \mathbb{F}$ as the closure of $\mathrm{Gr}_{G,\lambda,\mathbb{F}} \times_{\mathrm{Gr}_G} \mathrm{Gr}^\nabla$. Proposition 7.1 ensures that $\mathcal{C}_\lambda$ is an integral closed subscheme of dimension

$$\sum_{\text{positive } \alpha^\vee} \min\{e, \langle \alpha^\vee, \lambda \rangle\}$$

provided $\langle \alpha^\vee, \lambda \rangle \leq \mathrm{char}\, \mathbb{F} + e - 1$ for every positive root $\alpha^\vee$.

Proposition 8.3. *Assume that G admits a twisting element $\rho \in X_*(T)$ and suppose that $\mu = (\mu_1, \ldots, \mu_e)$ with $\mu_i \in X_*(T)$ strictly dominant. If $\mathrm{char}\, \mathbb{F} > 0$ assume also that*

$$\sum_i \langle \alpha^\vee, \mu_i \rangle \leq \mathrm{char}\, \mathbb{F} + e - 1$$

for every positive root $\alpha^\vee$. Then there exists $m_\lambda \in \mathbb{Z}$ so that as $e|R^+|$-dimensional cycles in $\mathrm{Gr}_G \otimes_{\mathcal{O}} \mathbb{F}$

$$[M_\mu \otimes_{\mathcal{O}} \mathbb{F}] = \sum_{\lambda'} m_\lambda [\mathcal{C}_{\lambda + e\rho}]$$

with the sum running over dominant $\lambda \leq \mu_1 + \ldots + \mu_e - e\rho$. Furthermore, $m_{\mu_1 + \ldots + \mu_e - e\rho} = 1$.

Later on we will give the m_λ a representation theoretic interpretation (see Theorem 12.1).

Proof. Propositions 6.8 and 5.9 ensures $M_\mu \otimes_{\mathcal{O}} \mathbb{F}$ factors through Gr_G^∇ and $Y_{G,\leq\mu}$. Lemma 5.6 implies $(Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F})_{\mathrm{red}} = \bigcup_{\lambda \leq \mu_1 + \ldots + \mu_e - e\rho} \mathrm{Gr}_{G,\lambda + e\rho,\mathbb{F}}$ and so

$$(M_\mu \otimes_{\mathcal{O}} \mathbb{F})_{\mathrm{red}} \subset \bigcup_{\lambda \leq \mu_1 + \ldots + \mu_e - e\rho} \mathrm{Gr}_{G,\lambda + e\rho,\mathbb{F}} \times_{\mathrm{Gr}_G} \mathrm{Gr}_G^\nabla \subset \bigcup_{\lambda \leq \mu_1 + \ldots + \mu_e - e\rho} \mathcal{C}_{\lambda + e\rho} \tag{8.4}$$

These unions run over λ which are not necessarily dominant. To show that the containment still holds with the union running over dominant λ we use the assumption that each μ_i is strictly dominant. This ensures that $\dim G/P_{\mu_i} = |R|$ and so $\dim M_\mu \otimes_{\mathcal{O}} \mathbb{F} = e|R^+|$. Thus

$$\dim \mathcal{C}_{\lambda + e\rho} = \sum_{\text{positive } \alpha^\vee} \min\{e, \langle \alpha^\vee, \lambda + e\rho \rangle\} \leq \dim M_\mu \otimes_{\mathcal{O}} \mathbb{F}$$

with equality if and only if $\langle \alpha^\vee, \lambda + e\rho \rangle \geq e$ for every positive $\alpha^\vee$. Notice that $\langle \alpha^\vee, \lambda + e\rho \rangle \geq e$ for every positive root $\alpha^\vee$ (equivalently every simple root) if and only if λ is dominant, because $\langle \alpha^\vee, \rho \rangle = 1$ whenever $\alpha^\vee$ is simple. Thus, (8.4) can be refined to:

$$(M_\mu \otimes_{\mathcal{O}} \mathbb{F})_{\mathrm{red}} \subset \bigcup \mathcal{C}_{\lambda + e\rho}$$

with the union running over dominant $\lambda \leq \mu_1 + \ldots + \mu_e - e\rho$. In other words,

$$[M_\mu \otimes_{\mathcal{O}} \mathbb{F}] = \sum_{\lambda'} m_\lambda [\mathcal{C}_{\lambda + e\rho}]$$

as desired. To finish the proof we have to $m_{\mu_1 + \ldots + \mu_e - e\rho} = 1$, and for this it suffices to show $\mathcal{C}_{\mu_1 + \ldots + \mu_e} \subset M_\mu \otimes_{\mathcal{O}} \mathbb{F}$ and this this closed immersion becomes an open immersion after restricting to an open subset of $\mathcal{C}_{\mu_1 + \ldots + \mu_e}$. Recall from Lemma 5.6 that $\mathrm{Gr}_{G,\mu_1 + \ldots + \mu_e,\mathbb{F}}$ is open in $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$. Therefore,

$$U := M_\mu \times_{\mathrm{Gr}_G} \mathrm{Gr}_{G,\mu_1 + \ldots + \mu_e,\mathbb{F}}$$

is open in $M_\mu \otimes_{\mathcal{O}} \mathbb{F}$. It is also non-empty because it is easy to see that $\mathcal{E}_{\mu_1 + \ldots + \mu_e,\mathbb{F}} \in M_\mu$. Therefore, U has dimension equal $\dim M_\mu \otimes_{\mathcal{O}} \mathbb{F}$. On the other hand U is a closed subscheme of $\mathrm{Gr}_{G,\mu_1+\ldots+\mu_e,\mathbb{F}} \times_{\mathrm{Gr}_G} \mathrm{Gr}^\nabla$. We saw in Proposition 7.1 that $\mathrm{Gr}_{G,\mu_1+\ldots+\mu_e,\mathbb{F}} \times_{\mathrm{Gr}_G} \mathrm{Gr}^\nabla$ is smooth and irreducible of the same dimension. Thus, $U = \mathrm{Gr}_{G,\mu_1+\ldots+\mu_e,\mathbb{F}} \times_{\mathrm{Gr}_G} \mathrm{Gr}^\nabla$. As $\mathcal{C}_{\mu_1+\ldots+\mu_e}$ is the closure of U we conclude that $\mathcal{C}_{\mu_1+\ldots+\mu_e} \subset M_\mu \otimes_{\mathcal{O}} \mathbb{F}$, and that this inclusion is an isomorphism over an open subset. □

9. Irreducibility

Theorem 9.1. *Assume that G contains a twisting element $\rho \in X_*(T)$ and suppose that $\lambda \in X_*(T)$ is dominant. If $\mathrm{char}\,\mathbb{F} > 0$ then assume also that*

$$\langle \alpha^\vee, \lambda + e\rho \rangle \leq \mathrm{char}\,\mathbb{F} + e - 1$$

for every positive root $\alpha^\vee$. Then, as cycles

$$[M_{(\lambda+\rho,\rho,\ldots,\rho)} \otimes_{\mathcal{O}} \mathbb{F}] = [\mathcal{C}_{\lambda+e\rho}]$$

In other words, $M_{(\lambda+\rho,\rho,\ldots,\rho)} \otimes_{\mathcal{O}} \mathbb{F}$ is irreducible and generically reduced.

In order to prove the theorem we need the following lemma:

Lemma 9.2. *Suppose $\mu = (\mu_1, \ldots, \mu_e)$ with each $\mu_i \in X^*(T)$ dominant. Then any $\mathbb{F}$-valued point of $M_\mu(\mathbb{F})$ can be expressed as*

$$(\mathcal{E}, g_1 u^{\mu_1} g_2 u^\eta)$$

for some $g_1 \in G, g_2 \in L^+G$ and $\eta \leq \mu_2 + \ldots + \mu_e$.

Proof. Write $\mathrm{Gr}_G^{(e-1)}$ for the affine grassmannian defined as in Definition 4.1, but with the e-tuple $(\pi_1, \ldots, \pi_e)$ replaced by the $e-1$-tuple $(\pi_2, \ldots, \pi_e)$. Thus, the A-points of $\mathrm{Gr}_G^{(e-1)}$ classify isomorphism classes of pairs $(\mathcal{E}, \iota)$ with $\mathcal{E}$ a G-torsor on $A[u]$ and ι a trivialisation over $\mathrm{Spec}\,A[u, \prod_{i=2}^e (u-\pi_i)^{-1}]$. We have a morphism

$$m_0 : G \times_{\mathcal{O}} \mathrm{Gr}_G^{(e-1)} \to \mathrm{Gr}_G$$

(whose dependence on μ_1 we suppress from the notation) given by $(g, \mathcal{E}, \iota) \mapsto (\mathcal{E}, g(u-\pi_1)^{\mu_1} \circ \iota)$. We will prove the lemma by showing that every closed point of M_μ is contained in the image of $G \times_{\mathcal{O}} Y^{(e-1)}_{G,\leq(\mu_2,\ldots,\mu_e)}$ (where $Y^{(e-1)}_{G,\leq(\mu_2,\ldots,\mu_e)} \subset \mathrm{Gr}_G^{(e-1)}$ is defined as in Definition 5.2) under m_0. Using Lemma 5.6 we see this gives the desired result.

First observe that, under the identifications of Lemma 4.5, m_0 induces a surjection

$$\left(G \times_{\mathcal{O}} \mathrm{Gr}_G^{(e-1)}\right) \otimes_{\mathcal{O}} E \to (G/P_{\mu_1} \times_{\mathcal{O}} \mathrm{Gr}_{G,2} \times_{\mathcal{O}} \ldots \times_{\mathcal{O}} \mathrm{Gr}_{G,e}) \otimes_{\mathcal{O}} E$$

This is the case because $(\mathcal{E}, \iota) \in \mathrm{Gr}_G \otimes_{\mathcal{O}} E$ is contained in the right-hand side if and only if there is a $g \in G$ such that $(u-\pi_1)^{-\mu} g \circ \iota$ extends to a trivialisation of $\mathcal{E}$ on $U_1 = \mathrm{Spec}\,A[u, \prod_{j\neq 1}(u-\pi_j)^{-1}]$. In particular, this means that any E-valued point of M_μ is mapped onto by some $(g, \mathcal{E}, \iota) \in G \otimes_{\mathcal{O}} \mathrm{Gr}_G^{(e-1)}$ under m_0.

We will be done if we can show that $(\mathcal{E}, \iota) \in Y^{(e-1)}_{G,\leq(\mu_2,\ldots,\mu_e)}$. Choose a representation $\rho : G \to \mathrm{GL}(V)$ of highest weight χ and let $\mathcal{E}^\rho \subset V \otimes_{\mathcal{O}} E[u, \prod_{i\neq 1}(u-\pi_i)^{-1}]$ correspond to the image of $(\mathcal{E}, \iota)$ under $\mathrm{Gr}_G^{(e-1)} \to \mathrm{Gr}_{\mathrm{GL}(V)}^{(e-1)}$. We have to show

$$\mathcal{E}^\rho \subset \prod_{i=2}^e (u-\pi_i)^{\langle -w_0(\chi), \mu_i \rangle} V \otimes_{\mathcal{O}} E[u] \tag{9.3}$$

Proposition 6.8 implies $(\mathcal{E}, g(u-\pi_1)^{\mu_1} \circ \iota) \in Y_{G,\leq\mu}$ and so

$$\begin{aligned}\rho(g)(u-\pi_1)^{\rho\circ\mu_1}(\mathcal{E}^\rho) \subset \prod_{i=1}^{e}(u-\pi_i)^{\langle -w_0(\chi),\mu_i\rangle} V \otimes_{\mathcal{O}} E[u] \Rightarrow \\ (u-\pi_1)^{\rho\circ\mu_1}(\mathcal{E}^\rho) \subset \prod_{i=1}^{e}(u-\pi_i)^{\langle -w_0(\chi),\mu_i\rangle} V \otimes_{\mathcal{O}} E[u] \Rightarrow \\ \mathcal{E}^\rho \subset \prod_{i=2}^{e}(u-\pi_i)^{\langle -w_0(\chi),\mu_i\rangle} V \otimes_{\mathcal{O}} E[u,(u-\pi_1)^{-1}]\end{aligned}$$

From this (9.3) follows, since we already know that $\mathcal{E}^\rho \subset V \otimes_{\mathcal{O}} E[u, \prod_{i\neq 1}(u-\pi_i)^{-1}]$. □

Proof of Theorem 9.1. In view of Proposition 8.3, the theorem will follow if we can show that $\mathcal{C}_{\lambda'+e\rho} \notin M_{(\lambda+\rho,\rho,\dots,\rho)} \otimes_{\mathcal{O}} \mathbb{F}$ for any dominant $\lambda' < \lambda$. We will do this by choosing, for each $\lambda' < \lambda$, an $\mathbb{F}$-valued point in $\mathcal{C}_{\lambda'+e\rho}$ which will not be contained in $M_{(\lambda+\rho,\rho,\dots,\rho)}$ because it cannot be expressed in the form described by Lemma 9.2.

Step 1. Fix a dominant $\lambda' < \lambda$ and consider

$$\mathcal{E} := \prod_{\alpha^\vee>0} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle\alpha^\vee,\lambda'+\rho\rangle})\mathcal{E}_{\lambda'+e\rho,\mathbb{F}} \tag{9.4}$$

for some $b_{\alpha^\vee} \in \mathbb{F}$. We claim that the $b_{\alpha^\vee}$ can be chosen so that

(1) $\mathcal{E} \in \mathcal{C}_{\lambda'+e\rho}$
(2) $b_{\alpha^\vee} \neq 0$ for all simple $\alpha^\vee$.

The calculations from Step 4 in the proof of Proposition 7.1 show that such $b_{\alpha^\vee}$ exist.

Step 2. Assume for a contradiction that $\mathcal{E} \in M_{(\lambda+\rho,\rho,\dots,\rho)}$. Lemma 9.2 implies

$$u^{-\lambda+\rho} g \prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle\alpha^\vee,\lambda'+\rho\rangle})\mathcal{E}_{\lambda'+e\rho,\mathbb{F}} \in \mathrm{Gr}_{G,\eta,\mathbb{F}}$$

for some $g \in G$ and some dominant $\eta \leq (e-1)\rho$. As $\lambda+\rho$ is dominant we have $u^{-\lambda-\rho}B^- u^{\lambda+\rho} \in L^+G$ for B^- the Borel opposite to B. Since also $G = \bigcup_{w\in W} B^- wU$ for $U \subset B$ the unipotent subgroup, we can assume that $g = wb$ for $w_1 \in W$ and $b \in U$. Thus

$$u^{-\lambda-\rho} w_1 b \prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle\alpha^\vee,\lambda'+\rho\rangle})\mathcal{E}_{\lambda'+e\rho,\mathbb{F}} \in \mathrm{Gr}_{G,\eta,\mathbb{F}} \tag{9.5}$$

Now recall from the proof of Proposition 7.1 the action of $\mathbb{G}_m$ on Gr_G via loop rotations. This induces an action of $L^+T \rtimes \mathbb{G}_m$ on Gr_G (where $\mathbb{G}_m$ acts on L^+T in this semi-direct product via $t\cdot x(u) = x(tu)$). Each $\xi \in X_*(T)$ induces a 1-parameter subgroup $\mathbb{G}_m \to L^+T \rtimes \mathbb{G}_m$ via $(t \mapsto \xi(t), t)$. The resulting action of $\mathbb{G}_m$ on Gr_G stabilises $\mathrm{Gr}_{G,\eta,\mathbb{F}}$ and, if η is strictly dominant, the fixed points are the $\mathcal{E}_{w(\eta),\mathbb{F}}$ for $w \in W$. As a consequence, if we take $\xi = \lambda+\rho$, then (9.5) gives

$$\lim_{t\to 0} t \cdot u^{-\lambda-\rho} w_1 b \prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle\alpha^\vee,\lambda'+\rho\rangle})\mathcal{E}_{\lambda'+e\rho,\mathbb{F}} = \mathcal{E}_{w_2(\eta),\mathbb{F}}$$

for some $w_2 \in W$. However

$$\begin{aligned} t\cdot u^{-\lambda-\rho} w_1 b \prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle\alpha^\vee,\lambda'+\rho\rangle})\mathcal{E}_{\lambda'+e\rho,\mathbb{F}} = \\ u^{-\lambda-\rho} w_1 b_1 \prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle\alpha^\vee,\lambda'+\rho\rangle} t^{\langle\alpha^\vee,\lambda'+\rho\rangle})\mathcal{E}_{\lambda'+e\rho,\mathbb{F}}\end{aligned}$$

and so, since $\langle \alpha^\vee, \lambda' + \rho \rangle > 0$ due to the dominance of λ', we deduce that

$$w_1 b \mathcal{E}_{\lambda' + e\rho, \mathbb{F}} = \mathcal{E}_{\lambda + \rho + w_2(\eta), \mathbb{F}}$$

Since $\lambda' + e\rho$ is strictly dominant it follows that the image of b in G/B^- is T-stable. As $U \to G/B^-$ is an open immersion we conclude that $b = 1$ and that $w_1(\lambda' + e\rho) = \lambda + \rho + w_2(\eta)$.

Step 4. Substituting $b = 1$ and $w_1(\lambda' + e\rho) = \lambda + \rho + w_2(\eta)$ into (9.5) gives

$$\prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle \alpha^\vee, w_3(\eta) - (e-1)\rho \rangle}) \mathcal{E}_{w_3(\eta), \mathbb{F}} \in \mathrm{Gr}_{G, \eta, \mathbb{F}} \tag{9.6}$$

for $w_3 = w_1^{-1} w_2$. Since $\langle \alpha^\vee, w_3(\eta) - (e-1)\rho \rangle \leq 0$ it follows that

$$\prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle \alpha^\vee, w_3(\eta) - (e-1)\rho \rangle}) \mathcal{E}_{w_3(\eta), \mathbb{F}} \in \mathrm{Gr}^{\eta}_{G, \mathbb{F}} \cap \mathrm{Gr}_{G, \eta, \mathbb{F}}$$

where $\mathrm{Gr}^{\eta}_{G,\mathbb{F}}$ is the opposite Schubert cell defined as the $L^- G$-orbit of $\mathcal{E}_{\eta, \mathbb{F}}$ where $L^- G$ is the group scheme with A-valued points given by $G(A[u^{-1}])$. It follows from [Zhu17, 2.3.3] that $\prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle \alpha^\vee, w_3(\eta) - (e-1)\rho \rangle}) \mathcal{E}_{w_3(\eta), \mathbb{F}}$ is contained in the G-orbit of $\mathcal{E}_{\eta, \mathbb{F}}$. In particular, $\prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle \alpha^\vee, w_3(\eta) - (e-1)\rho \rangle}) \mathcal{E}_{w_3(\eta), \mathbb{F}}$ is fixed under the action of $\mathbb{G}_m$ by loop rotations.

Step 5. For the final step notice that

$$\prod_{\alpha^\vee} x_{\alpha^\vee}(b_{\alpha^\vee} u^{\langle \alpha^\vee, w_3(\eta) - (e-1)\rho \rangle}) \mathcal{E}_{w_3(\eta), \mathbb{F}} = u^{w_3(\eta) - (e-1)\rho} b_0 \mathcal{E}_{(e-1)\rho}$$

where $b_0 = \prod_{\alpha^\vee > 0} x_\alpha(b_\alpha) \in B$. That this element is fixed by loop rotations is the same as saying that $u^{w_3(\eta) - (e-1)\rho} t^{w_3(\eta) - (e-1)\rho} b_0 u^{(e-1)\rho} \in u^{w_3(\eta) - (e-1)\rho} b_0 u^{(e-1)\rho} L^+ G$ for t the variable of $\mathbb{G}_m$. This implies that

$$\mathrm{Ad}(b_0^{-1}) D_\eta(t) \in \mathrm{Ad}(u^{(e-1)\rho}) \mathfrak{g}[[u]]$$

for D_η the derivative of $w_3(\eta) - (e-1)\rho : \mathbb{G}_m \to T$ at the identity. Notice that $\mathrm{Ad}(b_0^{-1}) D_\eta(t) \in \mathfrak{g}$ so we must actually have

$$\mathrm{Ad}(b_0^{-1}) D_\eta(t) \in \mathfrak{t} \oplus \bigoplus_{\alpha^\vee < 0} \mathfrak{g}_{\alpha^\vee}$$

(recall $\mathfrak{t} = \mathrm{Lie}((T))$). On the other hand, passing from the identity in [Jan03, II.1.3] to the Lie algebra and inducting shows that if $t_0 \in \mathfrak{t}$ then $\mathrm{Ad}(b_0^{-1}) t_0 - t_0$ can be expressed as a sum over $\gamma^\vee > 0$ of terms

$$dx_{\gamma^\vee}(b_{\gamma^\vee} d\gamma^\vee(t_0) + D_\gamma)$$

where $d\gamma^\vee$ denotes the derivate of $\gamma^\vee$ and D_γ is a $\mathbb{Z}$-linear combination of products of $b_{\alpha^\vee}$ for $0 < \alpha^\vee < \gamma^\vee$. Since $b_{\gamma^\vee} \neq 0$ for each simple $\gamma^\vee$ we must have $D_\eta(t) = 0$, i.e $w_3(\eta) = (e-1)\rho$. Thus $w_3 = 1$ (i.e. $w_2 = w_1$) and $\eta = (e-1)\rho$, and so $\lambda + \rho = w_1(\lambda' + \rho)$. Since both λ and λ' are dominant we must have $w_1 = 1$ and so $\lambda = \lambda'$. This contradicts the fact that $\lambda' < \lambda$ and finishes the proof. $\square$

10. Equivariant sheaves and their cycles

Our goal is now to give the coefficients appearing in Proposition 8.3 a representation theoretic meaning. To do this we will relate these cycle identities to relations between global sections of line bundles on the M_μ.

10.1. If X is a finite type $\mathbb{F}$-scheme and equipped with an action of the torus T, then we write $K_0^T(X)$ for the Grothendieck group of the category of T-equivariant coherent sheaves on X. If X is additionally proper over $\mathbb{F}$ then the Euler characteristic $[\mathcal{F}] \mapsto \sum_{i\geq 0}(-1)^i[H^i(X,\mathcal{F})]$ defines a homomorphism

$$\chi : K_0^T(X) \to R(T)$$

where $R(T) = \mathbb{Z}[X^*(T)]$ denotes the Grothendieck group of algebraic T-representations (in which multiplication is given by the tensor product). For $\alpha^\vee \in X_*(T)$ we write $e(\alpha^\vee)$ for its class in $R(T)$ and for $V \in R(T)$ we write $V_{\alpha^\vee} \in \mathbb{Z}$ for the multiplicity of $e(\alpha^\vee)$ in V.

Definition 10.2. Suppose that $d \geq 0$.

- Define $K_0^T(X)_{\leq d}$ as the Grothendieck group of the category of T-equivariant coherent sheaves on X with support of dimension $\leq d$.
- Let $V = (V_n)_{n\geq 0}$ be a sequence of elements in $R(T)$. We say V is polynomial of degree $\leq d$ if there exists a polynomial $P(x) \in \mathbb{Q}[x]$ of degree $\leq d$ so that
$$\sum_{\alpha^\vee \in X^*(T)} |V_{n,\alpha^\vee}| \leq P(n)$$
for all $n \geq 0$ (here $|\cdot|$ denotes the usual absolute value).

Notice that if each V_n is effective (i.e. is the class of a T-representation V_n in $R(T)$) then V is polynomial if and only if the dimensions of V_n are bounded by the value at n of a polynomial. However the above definition also allows us to extend this notion to elements which are not necessarily effective.

Remark 10.3. Note that there are homomorphisms $K_0^T(X)_{\leq d} \to K_0^T(X)$ which are not typically injective.

Lemma 10.4. *If $V = (V_n)_{n\geq 0}$ and $W = (W_n)_{n\geq 0}$ are polynomial of degree $\leq d$ then $(V_n + W_n)_{n\geq 0}$ and $(V_nW_n)_{n\geq 0}$ are also polynomial of degree $\leq d$.*

Proof. This is clear since $\sum_{\alpha^\vee} |V_{n,\alpha^\vee}+W_{n,\alpha^\vee}| \leq \sum_{\alpha^\vee}|V_{n,\alpha^\vee}|+\sum_{\alpha^\vee}|W_{n,\alpha^\vee}|$ and similarly $\sum_{\alpha^\vee} |\sum_{\beta^\vee+\gamma^\vee=\alpha^\vee} V_{n,\beta^\vee}W_{n,\gamma^\vee}| \leq \sum_{\beta^\vee,\gamma^\vee}|V_{n,\beta^\vee}||W_{n,\gamma^\vee}| = \left(\sum_{\beta^\vee}|V_{n,\beta^\vee}|\right)\left(\sum_{\gamma^\vee}|V_{n,\gamma^\vee}|\right)$. □

Lemma 10.5. *Suppose that X is a proper $\mathbb{F}$-scheme of finite type equipped with a T-equivariant ample line bundle $\mathcal{L}$ and $\mathcal{F} \in K_0^T(X)$ is contained in the image of $\operatorname{im} K_0^T(X)_{\leq d}$. Then*

$$\chi(\mathcal{F} \otimes [\mathcal{L}^{\otimes n}]) \in R(T)$$

is polynomial of degree $\leq d$.

Proof. Using Lemma 10.4 we can assume that $\mathcal{F}$ is the class of a T-equivariant sheaf $\mathcal{G}$ on X with support of dimension $\leq d$. Since $\mathcal{L}$ is ample $\chi([\mathcal{G}\otimes\mathcal{L}^{\otimes n}])$ equals the class of $H^0(X,\mathcal{G}\otimes\mathcal{L}^{\otimes n})$ in $R(T)$ for sufficiently large n. Since the dimension of $H^0(X, \mathcal{G} \otimes \mathcal{L}^{\otimes n})$ is for sufficiently large n the value at n of a polynomial $P(x)$ of degree equal the support of $\mathcal{G}$ it follows that for all $n \geq 0$

$$\sum_{\alpha^\vee} |\chi(\mathcal{G} \otimes \mathcal{L}^{\otimes n})_{\alpha^\vee}| = \sum_{\alpha^\vee} \chi(\mathcal{G} \otimes \mathcal{L}^{\otimes n})_{\alpha^\vee} \leq P(n) + C$$

for a constant $C >> 0$. □

Proposition 10.6. *Let $\mathcal{F}$ be a T-equivariant coherent sheaf on X with support of dimension $\leq d$ and let*

$$[\mathcal{F}] = \sum_Z n_Z [\mathcal{O}_Z] \in Z_d(X)$$

be the associated d-dimensional cycle. Assume that X is equipped with an ample T-equivariant line bundle $\mathcal{L}$ and that $n_Z > 0$ implies Z is T-stable. Then there are $q_Z \in \mathbb{Z}_{\geq 0}$ and $\theta^\vee_{Z,i} \in X^*(T)$ so that

$$\chi(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) - \sum_Z \left(\sum_{i=1}^{n_Z} e(\theta^\vee_{Z,i}) \chi(\mathcal{L}^{\otimes k - q_Z}|_Z) \right) \in R(T)$$

is polynomial of degree $< d$.

Proof. We induct on the number of $n_Z > 0$. If this is zero then the class of $\mathcal{F}$ has support of dimension $< d$ and so its class in $K_0^T(X)$ is contained in $\operatorname{im} K_0^T(X)_{\leq d-1}$. Lemma 10.5 therefore implies $\chi(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is polynomial of degree $< d$, and the proposition holds. Otherwise, write $\mathcal{I}_Z$ for the ideal sheaves corresponding to those Z with $n_Z > 0$. By assumption each such Z is T-stable and so $\mathcal{I}_Z^N$ is a T-equivariant coherent sheaf for any $N \geq 1$. For N sufficiently large the support of $\mathcal{I}_Z^N \mathcal{F}$ does not contain Z (see [Sta17, 0Y19]) and so the inductive hypothesis holds for $\mathcal{I}_Z^N \mathcal{F}$. By applying Lemma 10.5 to the identity

$$[\mathcal{F}] = [\mathcal{I}_Z^N \mathcal{F}] + [\mathcal{F}/\mathcal{I}_Z^N \mathcal{F}]$$

in $K_0^T(X)$ we see that the proposition will hold if it holds for $\mathcal{F}/\mathcal{I}_Z^N \mathcal{F}$.

Since $\mathcal{F}/I_Z^N \mathcal{F}$ has support contained in Z we are reduced to proving the proposition when the support of $\mathcal{F}$ is a single irreducible component. Thus we can assume $Z = X$ and write $[\mathcal{F}] = n[\mathcal{O}_X]$ in $Z_d(X)$. Since $\mathcal{L}$ is ample there is an integer $q \in \mathbb{Z}_{\geq 0}$ so that $\mathcal{F} \otimes \mathcal{L}^{\otimes q}$ is generated by global sections. This gives a T-equivariant surjection

$$V \otimes \mathcal{L}^{\otimes -q} \to \mathcal{F}$$

with $V = H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes q})$. Since T is abelian we can choose a T-equivariant filtration

$$\ldots \subset V_{j+1} \subset V_j \subset V_{j-1} \subset \ldots V_0 = V$$

with each graded piece of dimension one.

Claim. *There exists $j_1, \ldots, j_n \in \mathbb{Z}_{\geq 0}$ so that*

$$[\mathcal{F}] - \sum_{i=1}^n [V_{j_i}/V_{j_i+1} \otimes \mathcal{L}^{-q}] \in \operatorname{im} K_0^T(X)_{\leq d-1}$$

Proof of Claim. Set $\mathcal{F}_j$ equal the image in $\mathcal{F}$ of $V_j \otimes \mathcal{L}^{-q}$. Then $[\mathcal{F}] = \sum_{j \in \mathbb{Z}_{\geq 0}} [\mathcal{F}_j/\mathcal{F}_{j+1}]$ in $K_0^T(X)$. For each j we also have a T-equivariant exact sequence

$$0 \to \mathcal{G}_j \to V_j/V_{j+1} \otimes \mathcal{L}^{-q} \to \mathcal{F}_j/\mathcal{F}_{j+1} \to 0$$

of coherent sheaves. Since $V_j/V_{j+1}\otimes\mathcal{L}^{-q}$ is locally free of rank one it follows that exactly one of $\mathcal{G}_j$ and $\mathcal{F}_j/\mathcal{F}_{j+1}$ has support of dimension $<d$ and exactly one has support equal to X. Thus

$$\begin{aligned}[\mathcal{F}] - \sum_{\operatorname{supp}\mathcal{F}_j/\mathcal{F}_{j+1}=X}[V_j/V_{j+1}\otimes\mathcal{L}^{\otimes -q}] \\ = -\sum_{\operatorname{supp}\mathcal{F}_j/\mathcal{F}_{j+1}=X}[\mathcal{G}_j] + \sum_{\operatorname{supp}\mathcal{F}_j/\mathcal{F}_{j+1}\neq X}[\mathcal{F}_j/\mathcal{F}_{j+1}] \in \operatorname{im} K_0^T(X)_{\leq d-1}\end{aligned}$$

To finish the proof of the claim we just need to check that $\operatorname{supp}\mathcal{F}_j/\mathcal{F}_{j+1}=X$ exactly n times. This follows because in $Z_d(X)$ we have $[V_j/V_{j+1}\otimes\mathcal{L}^{\otimes -q}]=[X]$ and so $[\mathcal{F}]=\sum_{\operatorname{supp}\mathcal{F}_j/\mathcal{F}_{j+1}=X}[V_j/V_{j+1}\otimes\mathcal{L}^{\otimes -q}]=\sum_{\operatorname{supp}\mathcal{F}_j/\mathcal{F}_{j+1}=X}[X]=n[X]$. □

Applying Lemma 10.5 to the identity in the claim gives the proposition because if $\theta_i^\vee\in X^*(T)$ is the character through which T acts on V_{j_i}/V_{j_i+1} then $\chi(V_j/V_{j+1}\otimes\mathcal{L}^{\otimes k-q})=e(\theta_i^\vee)\chi(\mathcal{L}^{\otimes(k-q)})$. □

11. Determinant line bundles

11.1. In order to apply Proposition 10.6 to the identity of cycles established in Proposition 8.3 we need to choose an equivariant line bundle on Gr_G. To do this we consider the morphism

$$\mathrm{Ad}:\mathrm{Gr}_G\to\mathrm{Gr}_{\mathrm{GL}(\mathfrak{g})}$$

induced by the adjoint representation of G. Then $\mathrm{Gr}_{\mathrm{GL}(\mathfrak{g})}$ is equipped with the "determinantal" line bundle $\mathcal{L}_{\det}$, defined by the property that its pull-back to $\operatorname{Spec}A$ along a morphism corresponding to $(\mathcal{E},\iota)\in\mathrm{Gr}_{\mathrm{GL}(\mathfrak{g})}(A)$ is given by the A-module

$$\det_A(u^{-N}\mathfrak{g}\otimes_{\mathcal{O}}A[u]/\iota(\mathcal{E}))\otimes_A\det_A(u^{-N}\mathfrak{g}\otimes_{\mathcal{O}}A[u]/\mathfrak{g}\otimes_{\mathcal{O}}A[u])^{-1}$$

for N sufficiently large that $\iota(\mathcal{E})\subset u^{-N}\mathfrak{g}\otimes_{\mathcal{O}}A[u]$. Note this is independent of the choice of N. Then $\mathcal{L}_{\det}$ is $\mathrm{GL}(\mathfrak{g})$-equivariant and is ample in the sense that its restriction to any closed subscheme in $\mathrm{Gr}_{\mathrm{GL}(\mathfrak{g})}$ is ample. Therefore

$$\mathcal{L}_{\mathrm{ad}}\coloneqq\mathrm{Ad}^*\mathcal{L}_{\det}$$

is G-equivariant and also ample.

Lemma 11.2. *For $\lambda\in X_*(T)$ the group T acts on the fibre of $\mathcal{L}_{\mathrm{ad}}$ over $\mathcal{E}_{\lambda,i}$ via the image of λ under the homomorphism*

$$p:X_*(T)\to X^*(T)$$

given by $\lambda\mapsto\sum_{\alpha^\vee\in R^\vee}\langle\alpha^\vee,\lambda\rangle\alpha^\vee$.

Proof. This fibre is the rank one $\mathcal{O}$-module

$$\underbrace{\det_{\mathcal{O}}\left(u^{-N}\mathfrak{g}[u]/\operatorname{Ad}(u-\pi_i)^\lambda\mathfrak{g}[u]\right)}_{\Lambda_1}\otimes\underbrace{\det_{\mathcal{O}}\left(u^{-N}\mathfrak{g}[u]/\mathfrak{g}[u]\right)^{-1}}_{\Lambda_2}\tag{11.3}$$

for sufficiently large N. As an $\mathcal{O}$-module we have

$$u^{-N}\mathfrak{g}[u]/\operatorname{Ad}(u-\pi_i)^\lambda\mathfrak{g}[u]\cong\bigoplus_{\alpha^\vee\in R^\vee}\bigoplus_{n=-N}^{\langle\alpha^\vee,\lambda\rangle}(u-\pi_i)^n\mathfrak{g}_{\alpha^\vee}$$

and $t \in T(\mathcal{O})$ acts on $(u-\pi_i)^n \mathfrak{g}_{\alpha^\vee}$ by $\alpha^\vee(t)$. Therefore, t acts on Λ_1 by $\prod_{\alpha^\vee \in R} \alpha^\vee(t)^{\langle \alpha^\vee, \lambda \rangle + N}$. Similarly t acts on Λ_2 by $\prod_{\alpha^\vee \in R} \alpha^\vee(t)^N$. We conclude that t acts on (11.3) by

$$\prod_{\alpha^\vee \in R} \alpha^\vee(t)^{\langle \alpha^\vee, \lambda \rangle} = p(\lambda)(t)$$

as claimed. □

Proposition 11.4. *If $\mu = (\mu_1, \dots, \mu_e)$ with $\mu_i \in X_*(T)$ dominant then*

$$[H^0(M_\mu \otimes_{\mathcal{O}} \mathbb{F}, \mathcal{L}_{\mathrm{ad}}^{\otimes n})] = [\bigotimes_{i=1}^{e} \mathrm{Ind}_{P_{\mu_i}}^G (p(n\mu_i))] \in R(T)$$

for sufficiently large $n \geq 0$.

Proof. Since $\mathcal{L}_{\mathrm{ad}}$ is an ample line bundle on the flat $\mathcal{O}$-scheme M_μ it follows that

$$H^0(M_\mu \otimes_{\mathcal{O}} \mathbb{F}, \mathcal{L}_{\mathrm{ad}}^{\otimes n}) = H^0(M_\mu, \mathcal{L}_{\mathrm{ad}}^{\otimes n}) \otimes_{\mathcal{O}} \mathbb{F}$$

for sufficiently large n. Therefore $[H^0(M_\mu \otimes_{\mathcal{O}} \mathbb{F}, \mathcal{L}_{\mathrm{ad}}^{\otimes n})]$ is equal to the image of $[H^0(M_\mu \otimes_{\mathcal{O}} E, \mathcal{L}_{\mathrm{ad}}^{\otimes n})]$ under the specialisation map from [Jan03, 10.9]. Since this map sends the class of $\bigotimes_{i=1}^{e} \mathrm{Ind}_{P_{\mu_i}}^G (p(n\mu_i))$ viewed as a representation on an E-vector space onto $\bigotimes_{i=1}^{e} \mathrm{Ind}_{P_{\mu_i}}^G (p(n\mu_i))$ viewed as a representation on an $\mathbb{F}$-vector space the proposition will follow if we can show

$$[H^0(M_\mu \otimes_{\mathcal{O}} E, \mathcal{L}_{\mathrm{ad}}^{\otimes n})] = [\bigotimes_{i=1}^{e} \mathrm{Ind}_{P_{\mu_i}}^G (p(n\mu_i))] \in R(T)$$

Under the isomorphism $\mathrm{Gr}_G \otimes_{\mathcal{O}} E \cong (\mathrm{Gr}_{G,1} \times_{\mathcal{O}} \dots \times_{\mathcal{O}} \mathrm{Gr}_{G,e}) \otimes_{\mathcal{O}} E$ we have $\mathcal{L}_{\mathrm{ad}} = \bigotimes_{i=1}^{e} p_i^* \mathcal{L}_{\mathrm{ad},i}$ where $\mathcal{L}_{\mathrm{ad},i}$ is the restriction of $\mathcal{L}_{\mathrm{ad}}$ to $\mathrm{Gr}_{G,i}$ and p_i is the i-th projection. Therefore

$$\mathcal{L}_{\mathrm{ad}}|_{M_\mu \otimes_{\mathcal{O}} E} = \bigotimes_{i=1}^{e} p_i^* (\mathcal{L}_{\mathrm{ad,i}}|_{G/P_{\mu_i} \times_{\mathcal{O}} E})$$

and so the Kunneth formula [Sta17, 0BED] identifies

$$H^0(M_\mu \otimes_{\mathcal{O}} E, \mathcal{L}_{\mathrm{ad}}^{\otimes n}) = \bigotimes_{i=1}^{e} H^0(G/P_{\mu_i} \otimes_{\mathcal{O}} E, \mathcal{L}_{\mathrm{ad},i}^{\otimes n})$$

as G-representations. To finish the proof we just have to show

$$H^0(G/P_{\mu_i}, \mathcal{L}_{\mathrm{ad},i}^{\otimes n}) \cong \mathrm{Ind}_{P_{\mu_i}}^G (p(n\mu_i)) \tag{11.5}$$

For this recall (see for example [Jan03, 5.12]) that the global sections of any G-equivariant line bundle on G/P_{μ_i} are G-equivariantly isomorphic $\mathrm{Ind}_{P_{\mu_i}}^G(\eta)$ where $\eta \in X^*(T)$ is the character through which T acts on the fibre over $1 \in G/P_{\mu_i}$. Since 1 is mapped onto $\mathcal{E}_{\mu_i,i}$ under the closed immersion $G/P_{\mu_i} \to \mathrm{Gr}_{G,i}$ we deduce (11.5) from Lemma 11.2. □

12. Main theorem

For any dominant $\lambda^\vee \in X^*(T)$ write

$$W(\lambda^\vee) = \mathrm{Ind}_{B^-}^G (\lambda^\vee)$$

for B^- the Borel opposite to B. Likewise, for $\lambda \in X_*(T)$ we make sense of $W(\lambda)$, now as a representation of $\widehat{G}$. The following is an alternative formulation of Theorem 1.1.

Theorem 12.1. *Assume that G admits a twisting element $\rho \in X_*(T)$ and let $\mu = (\mu_1, \ldots, \mu_e)$ with $\mu_i \in X_*(T)$ strictly dominant. If $\operatorname{char}\mathbb{F} > 0$ assume also that*

$$\sum_{i=1}^{e} \langle \alpha^\vee, \mu_i \rangle \leq \operatorname{char}\mathbb{F} + e - 1$$

for all positive roots $\alpha^\vee$. Then

$$[M_\mu \otimes_{\mathcal{O}} \mathbb{F}] = \sum m_\lambda [M_{(\lambda+\rho, \ldots, \rho)}]$$

as $e|R^+|$-dimensional cycles for $m_\lambda \in \mathbb{Z}_{\geq 0}$ determined by the identity

$$[\bigotimes_{i=1}^{e} W(\mu_i - \rho)] = \sum_\lambda m_\lambda [W(\lambda)]$$

in the Grothendieck group of $\widehat{G}$-representations.

Proof. We give the proof here using two representation theoretic propositions from the next section (Propositions 13.2 and 13.4). Proposition 8.3 and Theorem 9.1 imply

$$[M_\mu \otimes_{\mathcal{O}} \mathbb{F}] = \sum n_\lambda [M_{\widetilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}]$$

where the sum suns over dominant $\lambda \leq \mu_1 + \ldots + \mu_e - e\rho$, $\widetilde{\lambda} = (\lambda + \rho, \rho, \ldots, \rho)$, and $n_\lambda \in \mathbb{Z}_{\geq 0}$. We have to show that $n_\lambda = m_\lambda$. Applying Proposition 10.6 to this identity with $\mathcal{L}$ equal to $\mathcal{L}_{\mathrm{ad}}$ gives $\theta^\vee_{\lambda,i} \in X^*(T)$ so that

$$\chi(\mathcal{L}_{\mathrm{ad}}^{\otimes n}|_{M_\mu \otimes_{\mathcal{O}} \mathbb{F}}) - \sum_\lambda \sum_{i=1}^{n_\lambda} e(\theta^\vee_{\lambda,i}) \chi(\mathcal{L}_{\mathrm{ad}}^{\otimes n}|_{M_{\widetilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}})$$

is polynomial of degree $< e|R^\vee_+|$. Since each μ_i is strictly dominant each P_{μ_i} equals the opposite Borel B^- and so

$$W(p(n\mu_i)) = \operatorname{Ind}_{P_{\mu_i}}^{G}(p(n\mu_i))$$

Therefore Proposition 11.4 gives that

$$\prod_{i=1}^{e} W(p(n\mu_i)) - \sum_\lambda \sum_{i=1}^{n_\lambda} e(\theta^\vee_{\lambda,i}) W(p(n(\lambda+\rho))) W(p(n\rho))^{e-1}$$

is polynomial of degree $< e|R^\vee_+|$. In the next section we prove (see Proposition 13.2) that

$$\prod_{i=1}^{e} W(p(n\mu_i)) - \sum_\lambda m_\lambda W(p(n(\lambda+\rho))) W(p(n\rho))^{e-1}$$

is polynomial of degree $< e|R^\vee_+|$, and so considering the difference gives that

$$\sum_\lambda \left(m_\lambda - \sum_{i=1}^{n_\lambda} e(\theta^\vee_{\lambda,i}) \right) W(p(n(\lambda+\rho))) W(p(n\rho))^{e-1}$$

is also polynomial of degree $< e|R^\vee_+|$. In the next section we also prove (see Proposition 13.4) that if $X_\lambda \in R(T)$ are such that

$$\sum_\lambda X_\lambda W(p(n(\lambda+\rho))) W(p(n\rho))^{e-1}$$

is polynomial of degree $< e|R^\vee_+|$ then $X_\lambda = 0$ for each λ. Therefore

$$m_\lambda - \sum_{i=1}^{n_\lambda} e(\theta^\vee_{\lambda,i}) = 0$$

for each λ. This implies $n_\lambda = m_\lambda$ for each λ which finishes the proof. $\square$

13. Some representation theory

It remains to prove Proposition 13.2 and Proposition 13.4 which were used in the proof of Theorem 9.1. For this set $\rho^\vee = \frac{1}{2}\sum_{\alpha^\vee \in R_+^\vee} \alpha^\vee$. Then the Weyl character formula asserts that for any dominant $\lambda^\vee \in X^*(T)$

$$[W(\lambda^\vee)] = \frac{A(\lambda^\vee + \rho^\vee)}{A(\rho^\vee)}$$

where $A(\lambda^\vee) := \sum_{w \in W}(-1)^{l(w)} e(w(\lambda^\vee))$ and this identity is occurring inside the ring $\mathbb{Z}[\frac{1}{2}X^*(T)]$. See, for example, [Jan03, 5.10].

Lemma 13.1. *If $\mu \in X^*(T)$ is strictly dominant then*

$$W(n\mu^\vee) - e(\rho^\vee)\frac{A(n\mu^\vee)}{A(\rho^\vee)} \in R(T)$$

is a sequence of effective elements (i.e. $\mathbb{Z}_{\geq 0}$-linear combinations of the $e(\alpha^\vee)$) which is polynomial in n of degree $< |R^+|$.

Proof. We first reduce to the case where $X^*(T)$ contains a twisting element $\rho_0^\vee$, in the sense of Definition 2.2. The construction from [BG14, §5.3] produces a central extension $1 \to \mathbb{G}_m \to \widetilde{G} \to G \to 1$ such that if $\widetilde{T} \subset \widetilde{G}$ is the preimage of T then $X^*(\widetilde{T})$ contains such a twisting element. Being a central extension, the Weyl group of $\widetilde{G}$ relative to $\widetilde{T}$ equals W. Therefore, the inclusion

$$X^*(T) \to X^*(\widetilde{T})$$

maps $A(\lambda^\vee)$ onto $A(\widetilde{\lambda}^\vee)$ for $\widetilde{\lambda}^\vee$ the character of $\widetilde{T}$ induced by $\lambda^\vee$. As a result the lemma holds for G if it holds for $\widetilde{G}$

We can therefore assume there exists a twisting element $\rho_0^\vee \in X^*(T)$. Then $\rho_0^\vee - \rho^\vee$ is W-invariant and so the Weyl character formula implies $[W(\mu^\vee)] = \frac{A((\mu^\vee + \rho_0^\vee))}{A(\rho_0^\vee)}$. Now, for any $\lambda^\vee \in X^*(T)$ write $\mathcal{L}(\lambda^\vee)$ for the G-equivariant line bundle on G/B^- on which the action of T on the fibre over the identity is given by $\lambda^\vee$. Then $\mathcal{L}(\rho_0^\vee)$ admits a unique global section on which T acts by $\rho_0^\vee$, and this induces a T-equivariant injection

$$\mathcal{O}_{G/B^-} \otimes \rho_0^\vee \hookrightarrow \mathcal{L}_{\rho_0^\vee}$$

Tensoring with $\mathcal{L}_{\mu^\vee - \rho_0^\vee}$ produces a T-equivariant injection $\mathcal{L}_{\mu^\vee - \rho_0^\vee} \otimes \rho_0^\vee \hookrightarrow \mathcal{L}_{\mu^\vee}$ and taking global sections yields a T-equivariant injection of $W(\mu^\vee - \rho_0^\vee) \otimes \rho_0^\vee$ into $W(\mu^\vee)$. In particular,

$$[W(n\mu^\vee)] - e(\rho_0^\vee)[W(n\mu^\vee - \rho_0^\vee)] = [W(n\mu^\vee)] - e(\rho_0^\vee)\frac{A(n\mu^\vee)}{A(\rho_0^\vee)} = [W(n\mu^\vee)] - e(\rho^\vee)\frac{A(n\mu^\vee)}{A(\rho^\vee)}$$

is an effective element of $R(T)$ for each $n \geq 0$. Since it is effective we can show the sequence of elements is polynomial of degree $< |R^+|$ by showing that the difference between the dimensions of $W(n\mu^\vee)$ and $W(n\mu^\vee - \rho_0^\vee)$ is a polynomial in n of degree $< |R^+|$. But this follow from the Weyl dimension formula $\dim W(\lambda^\vee) = \prod_{\alpha \in R_+} \frac{\langle \lambda^\vee + \rho_0^\vee, \alpha \rangle}{\langle \rho_0^\vee, \alpha \rangle}$ since it shows both $W(n\mu^\vee)$ and $W(n\mu^\vee - \rho_0^\vee)$ have dimension the value at n of a degree $|R^+|$ polynomial with leading term $n^{|R^+|} \prod_{\alpha \in R_+} \frac{\langle \mu^\vee, \alpha \rangle}{\langle \rho_0^\vee, \alpha \rangle}$. $\square$

Proposition 13.2. *Suppose that $X_*(T)$ contains a twisting element ρ and consider strictly dominant $\mu_1,\dots,\mu_e \in X_*(T)$. If*

$$[\bigotimes_{i=1}^{e} W(\mu_i-\rho)] = \sum_\lambda m_\lambda [W(\lambda)]$$

in the Grothendieck group of $\widehat{G}$-representations then

$$\prod_{i=1}^{e} W(p(n\mu_i)) - \sum_\lambda m_\lambda W(p(n(\lambda+\rho)))W(p(n\rho))^{e-1}$$

is polynomial of degree $< e|R^\vee_*|$.

Proof. The Weyl character formula (applied to $\widehat{G}$) yields the identity

$$\prod_{i=1}^{e} \frac{A(\mu_i)}{A(\rho)} = \sum_\lambda m_\lambda \frac{A(\lambda+\rho)}{A(\rho)}$$

in $R(\widehat{T})$. Multiplying by $A(\rho)^e$ gives

$$\prod_{i=1}^{e} A(\mu_i) = \sum_\lambda m_\lambda A(\lambda+\rho)A(\rho)^{e-1}$$

The endomorphism of $R(\widehat{T}) = \mathbb{Z}[X_*(T)]$ induced by multiplication by n on $X^*(T)$ is W-equivariant and so commutes with the formation of $A(\lambda)$. Applying this endomorphism to the previous identity gives

$$\prod_{i=1}^{e} A(n\mu_i) = \sum_\lambda m_\lambda A(m(\lambda+\rho))A(n\rho)^{e-1}$$

The homomorphism $p: X_*(T) \to X^*(T)$ induces a homomorphism $R(T^\vee) \to R(T)$ which is again W-equivariant and so also commutes with the formation of $A(\lambda)$. Therefore, applying this homomorphism and multiplying by $\left(\frac{e(\rho^\vee)}{A(\rho^\vee)}\right)^e$ gives

$$\prod_{i=1}^{e} \frac{e(\rho^\vee)A(np(\mu_i))}{A(\rho^\vee)} = \sum_\lambda m_\lambda \frac{e(\rho^\vee)A(np(\lambda+\rho))}{A(\rho^\vee)} \frac{e(\rho^\vee)A(np(\rho))}{A(\rho^\vee)}^{e-1} \tag{13.3}$$

in $R(T)$. Write

$$\begin{aligned}\prod_{i=1}^{e} \frac{e(\rho^\vee)A(np(\mu_i))}{A(\rho^\vee)} &= \prod_{i=1}^{e}\left(W(np(\mu_i)) - \left(W(np(\mu_i)) - \frac{e(\rho^\vee)A(np(\mu_i))}{A(\rho^\vee)}\right)\right)\\ &= \prod_{i=1}^{e}(W(np(\mu_i))) + C_{\mu,n}\end{aligned}$$

for $C_{\mu,n} \in R(T)$. Lemma 13.1 ensures that $\left(W(np(\mu_i)) - \frac{e(\rho^\vee)A(np(\mu_i))}{A(\rho^\vee)}\right)$ is polynomial in n of degree $< |R^+|$ and so, since $W(np(\mu_i))$ has dimension polynomial in n of degree $|R^\vee_+|$, it follows that $C_\mu = (C_{\mu,n})_{n\geq 0}$ is polynomial of degree $< e|R^\vee_+|$. Similarly, each

$$\frac{e(\rho^\vee)A(np(\lambda+\rho))}{A(\rho^\vee)} \frac{e(\rho^\vee)A(np(\rho))}{A(\rho^\vee)}^{e-1} = W(np(\lambda+\rho))W(np(\rho))^{e-1} + C_{\lambda,n}$$

with $C_\lambda = (C_{\lambda,n})_{n\geq 0}$ polynomial of degree $< e|R^\vee_+|$. Combining these observations with (13.3) gives that

$$\prod_{i=1}^{e} W(p(n\mu_i)) - \sum_\lambda m_\lambda W(p(n(\lambda+\rho)))W(p(n\rho))^{e-1}$$

is polynomial of degree $< e|R_+^\vee|$ as desired. $\square$

Proposition 13.4. *Suppose that for strictly dominant $\lambda^\vee, \mu^\vee \in X^*(T)$ there are $C_\lambda \in R(T)$ such that*

$$\sum_\lambda C_{\lambda^\vee} W(n\lambda^\vee) W(n\mu^\vee)^{e-1}$$

is polynomial of degree $< e|R_+^\vee|$. Then $C_\lambda = 0$ for all λ.

Proof. Since $\mu^\vee$ is strictly dominant the dimension of $W^0(n\mu^\vee)$ is polynomial in n of degree $|R^+|$. Therefore, we can assume $e = 1$. If the proposition does not hold then we can choose a λ with $C_{\lambda^\vee} \neq 0$ so that $C_{\lambda_0^\vee} \neq 0$ implies $\lambda_0^\vee \leq \lambda^\vee$.

Observation. Let $\Phi, \Psi > 0$ be and let $S^\vee$ denote the set of simple roots. Then there exists a degree $|R_+^\vee|$ polynomial $Q(x) \in \mathbb{Q}[x]$ with positive leading term so that for $n \geq \Psi$ one has

$$\sum_{\eta^\vee} \dim W(n\lambda^\vee)_{\eta^\vee} \geq Q(n)$$

where the sum runs over $\eta^\vee = n\lambda^\vee - \sum_{\alpha^\vee \in S^\vee} l_{\alpha^\vee}\alpha^\vee$ with $0 \leq l_{\alpha^\vee} < \frac{n}{\Psi} - \Phi$.

Proof of Observation. The Kostant multiplicity formula [?] asserts that

$$\dim W(n\lambda^\vee)_{\eta^\vee} = \sum_{w \in W} (-1)^{l(w)} P(w(n\lambda^\vee + \rho^\vee) - (\eta^\vee + \rho^\vee))$$

where $P(\mu^\vee)$ denotes the number of ways in which $\mu^\vee \in X^*(T)$ can be expressed as a $\mathbb{Z}_{\geq 0}$-linear combination of $\alpha^\vee \in R_+^\vee$. We claim $P(w(n\lambda^\vee + \rho^\vee) - (\eta^\vee + \rho^\vee)) = 0$ for $w \neq 1$. Since

$$w(n\lambda^\vee + \rho^\vee) - (\eta^\vee + \rho^\vee) = w(n\lambda^\vee + \rho^\vee) - (n\lambda^\vee + \rho^\vee) + \sum_{\alpha^\vee \in S^\vee} l_{\alpha^\vee}\alpha^\vee$$

the claim follows if, when $w(n\lambda^\vee + \rho^\vee) - (n\lambda^\vee + \rho^\vee)$ is expressed as a $\mathbb{Z}$-linear combination of $\alpha^\vee \in S^\vee$, at least one coefficient is $\leq -n$. But this is clear since $\lambda^\vee$ is dominant (see, for example, [Hum78, 13.2.A]). Therefore the observation is reduced to producing a polynomial lower bound on

$$\sum_{0 \leq l_{\alpha^\vee} < \frac{n}{\Psi} - \Phi} P\Big(\sum_{\alpha^\vee \in S^\vee} l_{\alpha^\vee}\alpha^\vee\Big) \tag{13.5}$$

of the correct degree. To do this we first claim that

$$P\Big(\sum_{\alpha^\vee \in S^\vee} l_{\alpha^\vee}\alpha^\vee\Big) \geq \left(\frac{1}{|R_+^\vee| - |S^\vee|} \min\{l_{\alpha^\vee}\}\right)^{|R_+^\vee| - |S^\vee|}$$

This can be seen by noticing that if $0 \leq j_{\alpha^\vee} \leq \frac{1}{|R_+^\vee| - |S^\vee|} \min\{l_{\alpha^\vee}\}$ for $\alpha^\vee \in R_+^\vee \setminus S^\vee$ then there exists $i_{\alpha^\vee} \geq 0$ for $\alpha^\vee \in S^\vee$ so that

$$\sum_{\alpha^\vee \in S^\vee} l_{\alpha^\vee}\alpha^\vee = \sum_{\alpha^\vee \in R_+^\vee \setminus S^\vee} j_{\alpha^\vee}\alpha^\vee + \sum_{\alpha^\vee \in S^\vee} i_{\alpha^\vee}\alpha^\vee$$

(indeed every $\alpha^\vee \in R_+^\vee \setminus S^\vee$ can be expressed as a sum of $\alpha^\vee \in S^\vee$ and so $\sum_{\alpha^\vee \in R_+^\vee \setminus S^\vee} j_{\alpha^\vee}\alpha^\vee$ can be expressed as a linear combination of $\alpha^\vee \in S^\vee$ with the $\alpha^\vee$-coefficient in the interval $[0, \min\{l_{\alpha^\vee}\}]$). Therefore (13.5) is

$$\geq \sum_{0 \leq l_\alpha < \frac{k}{\Psi} - \Phi} \left(\frac{1}{|R_+^\vee| - |S^\vee|} \min\{l_{\alpha^\vee}\}\right)^{|R_+^\vee| - |S^\vee|}$$

which is easily seen to be a polynomial in n of degree $(|R_+^\vee|-|S^\vee|)+|S^\vee|$ with positive leading term. $\square$

We return to the proof of the proposition. Choose $\Psi, \Phi > 0$ (we will be more specific later). If $e(\theta^\vee)$ appears in $C_{\lambda^\vee}$ with non-zero multiplicity then $e(\theta^\vee + n\lambda^\vee - \sum_{\alpha^\vee} l_{\alpha^\vee}\alpha^\vee)$ appears in $C_{\lambda^\vee}W(n\lambda^\vee)$ for any n and any $0 \leq l_{\alpha^\vee} < \frac{n}{\Phi} - \Phi$. The observation above implies that for $n >> 0$ at least one of these $e(\theta^\vee + n\lambda^\vee - \sum_{\alpha^\vee} l_{\alpha^\vee}\alpha^\vee)$ must cancel in $\sum_{\lambda_0^\vee} C_{\lambda_0^\vee} W(n\lambda_0^\vee)$, since otherwise we contradict that assumption that $\sum_{\lambda_0^\vee} C_{\lambda_0^\vee} W(n\lambda_0^\vee)$ is polynomial in n of degree $< |R_+^\vee|$. Therefore, for each sufficiently large n there exists $0 \leq l_{\alpha^\vee} < \frac{n}{\Psi} - \Phi$ and $e(\theta_0^\vee)$ appearing with non-zero multiplicity in $C_{\lambda_0^\vee}$ for $\lambda_0^\vee \neq \lambda^\vee$ so that $e(\theta^\vee + n\lambda^\vee - \sum_{\alpha^\vee} l_{\alpha^\vee}\alpha^\vee)$ appears in $e(\theta_0^\vee)W(n\lambda_0^\vee)$. This implies

$$n\lambda^\vee - \sum_{\alpha^\vee} l_{\alpha^\vee}\alpha^\vee \leq \theta_0^\vee - \theta^\vee + n\lambda_0^\vee$$

Choose $\beta^\vee \in X^*(T)$ so that $\alpha^\vee \in S^\vee$ and the $\beta^\vee$ form a basis of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. If $\theta_0^\vee - \theta^\vee = \sum_{\alpha^\vee \in S^\vee} n_{\alpha^\vee}\alpha^\vee + \sum_{\beta^\vee} n_{\beta^\vee}\beta^\vee$ then

$$n(\lambda_0^\vee - \lambda^\vee) = \sum_{\alpha^\vee \in S^\vee} m_{\alpha^\vee,n}\alpha^\vee + \sum_{\beta^\vee} n_{\beta^\vee}\beta^\vee$$

with $m_{\alpha^\vee,n} \geq n_{\alpha^\vee} - l_{\alpha^\vee}$. Since the $n_{\beta^\vee}$ are bounded above independently of n (as there are only finitely many possible $\theta_0^\vee$) it follows that each $n_{\beta^\vee} = 0$. If $\Phi \geq -n_{\alpha^\vee}$ for every $\alpha^\vee$ then we also have $m_{\alpha^\vee,n} \geq -\frac{n}{\Psi}$. Therefore

$$\lambda_0^\vee - \lambda^\vee = \sum_{\alpha^\vee \in S^\vee} m_{\alpha^\vee}\alpha^\vee$$

with $m_{\alpha^\vee} \geq -\frac{1}{\Psi}$ for all $\Psi > 0$. We conclude that each $m_{\alpha^\vee} \geq 0$ and so $\lambda_0^\vee \geq \lambda^\vee$. Since this contradicts the maximality of $\lambda^\vee$ we conclude $C_{\lambda^\vee} = 0$ for every $\lambda^\vee$. $\square$

Part 2. Cycle identities in moduli spaces of crystalline representations

14. Notation

14.1. For the second part of this paper we fix the following data:

- Let $K/\mathbb{Q}_p$ be a finite extension with residue field k and ramification degree e over $\mathbb{Q}_p$. Let C denote a completed algebraic closure of K with ring of integers $\mathcal{O}_C$ and fix a compatible system π^{1/p^∞} of p-th power roots of a uniformiser $\pi \in K$.
- Fix another extension E of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$, and an embedding $k \hookrightarrow \mathbb{F}$ which we extend to an embedding $W(k) \to \mathcal{O}$. Enlarging E if necessary we assume that E contains a Galois closure of K so that $W(k) \hookrightarrow \mathcal{O}$ extends to e distinct embeddings, which we index as $\kappa_1, \ldots, \kappa_e$.
- Let G be a split reductive group over $\mathcal{O}$. Unlike in Part 1, we assume additionally that G has connected fibres. We set
$$\widetilde{G} = \mathrm{Res}_{W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}/\mathcal{O}}\left(G \otimes_{\mathbb{Z}_p} W(k)\right)$$
(thus $\widetilde{G}(A) = G(W(k) \otimes_{\mathbb{Z}_p} A)$ for any $\mathcal{O}$-algebra A). Since $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O} = \prod_{i=1}^f \mathcal{O} \otimes_{W(k),\varphi^i} W(k)$, for $f = [k : \mathbb{F}_p]$ and φ the lifting to $W(k)$ of the p-th power map on k, we can also write
$$\widetilde{G} \cong \prod_{i=1}^{f} G \otimes_{W(k),\varphi^i} W(k)$$
We apply the constructions from Definition 4.1 to $\widetilde{G}$ and with $\pi_i := \kappa_i(\pi)$ to obtain the ind-scheme $\mathrm{Gr}_{\widetilde{G}}$. Notice we also have:
$$\mathrm{Gr}_{\widetilde{G}} \cong \prod_{i=1}^{f} \mathrm{Gr}_G \otimes_{W(k),\varphi^i} W(k)$$
Maintaining the notation from Part 1, we write $E(u) = \prod_{i=1}^e (u - \pi_i)$. Notice this coincides with the minimal polynomial of π in $W(k)[u]$.
- For any p-adically complete $\mathcal{O}$-algebra A we set $\mathfrak{S}_A := (W(k) \otimes_{\mathbb{Z}_p} A)[[u]]$ and equip this ring with the A-linear Frobenius φ sending $u \mapsto u^p$ and lifting the p-th power map on k. We frequently identify
$$G(\mathfrak{S}_A) = \widetilde{G}(A[[u]]) = \prod_{i=1}^{f} G(A[[u]])$$
and notice that the endomorphism of $G(\mathfrak{S}_A)$ induced by φ on $\mathfrak{S}_A$ identifies with the automorphism of $\prod_{i=1}^f G(A[[u]])$ given by $(g_i)_i \mapsto (\varphi'(g_i))_{i+1}$ where the i are viewed modulo f and φ' is the automorphism of $G(A[[u]])$ induced by the A-linear endomorphism of $A[[u]]$ given by $u \mapsto u^p$
- For any p-adically complete $\mathcal{O}$-algebra we also consider
$$A_{\mathrm{inf},A} := \varprojlim_a \varprojlim_i (W(\mathcal{O}_{C^\flat})/p^a \otimes_{\mathbb{Z}_p} A)/u^i$$
where $\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p$ and $u = [(\pi, \pi^{1/p}, \pi^{1/p^2}, \ldots)] \in W(\mathcal{O}_{C^\flat})$. We view $A_{\mathrm{inf},A}$ as an $\mathfrak{S}_A$-algebra via u and note that the lift of Frobenius on $W(\mathcal{O}_{C^\flat})$ induces a Frobenius φ on $A_{\mathrm{inf},A}$ which is compatible with that

on $\mathfrak{S}_A$. The natural G_K-action on $\mathcal{O}_C$ also induces a continuous (for the (u,p)-adic topology) G_K-action on $A_{\mathrm{inf},A}$ commuting with φ. We also have

$$W(C^\flat)_A := \varprojlim_a A_{\mathrm{inf},A}[\tfrac{1}{u}]/p^a$$

If A is topologically of finite type (i.e. $A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is of finite type) then $\mathfrak{S}_A \to A_{\mathrm{inf},A}$ is faithfully flat (in particular injective) [EG23, 2.2.13]. We only consider the $A_{\mathrm{inf},A}$ and $W(C^\flat)_A$ for such A.

- Fix a compatible system $\epsilon = (\epsilon_1, \epsilon_2, \dots)$ of primitive p-th power roots of unity in C. Then we can view $\epsilon \in \mathcal{O}_{C^\flat}$ and we set $\mu = [\epsilon] - 1$.
- A Hodge type μ for K is a tuple of (conjugacy classes) of cocharacters of G, indexed by the embeddings of K into $\overline{\mathbb{Q}}_p$ (although we always ignore that we are considering these cocharacters up to conjugacy). Since every such embedding factors through E we can (and typically do) interpret a Hodge type as an e-tuple of cocharacters of $\widetilde{G}$.

15. Moduli of Breuil–Kisin modules

15.1. For any p-adically complete $\mathcal{O}$-algebra A a G-Breuil–Kisin module (usually we omit the G-) over A is a G-torsor $\mathfrak{M}$ on $\operatorname{Spec} \mathfrak{S}_A$ equipped with an isomorphism

$$\varphi_{\mathfrak{M}} : \varphi^* \mathfrak{M}[\tfrac{1}{E(u)}] \xrightarrow{\sim} \mathfrak{M}[\tfrac{1}{E(u)}]$$

We refer to $\varphi_{\mathfrak{M}}$ as the Frobenius on $\mathfrak{M}$ and frequently write φ instead of $\varphi_{\mathfrak{M}}$ when there is no risk of confusion.

- Let $Z_G(A)$ be the category of Breuil–Kisin modules over A whose morphisms are isomorphisms of G-torsors compatible with the Frobenii.
- Let $\widetilde{Z}_G(A)$ be the category of pairs $(\mathfrak{M}, \iota)$ with $\mathfrak{M}$ a Breuil–Kisin module over A and ι a trivialisation of $\mathfrak{M}$ over $\operatorname{Spec} \mathfrak{S}_A$. Morphisms are isomorphisms of G-torsors compatible with the Frobenii and commuting with the trivialisation.

Any homomorphism of p-adically complete $\mathcal{O}$-algebras $A \to B$ induces a homomorphism $\mathfrak{S}_A \to \mathfrak{S}_B$ and pull back induces functors $Z_G(B) \to Z_G(A)$ and $\widetilde{Z}^N_G(B) \to \widetilde{Z}^N_G(A)$ making Z_G and $\widetilde{Z}_G$ into categories fibred over $\operatorname{Spf} \mathcal{O}$. In the obvious way these constructions are functorial in G.

Remark 15.2. If $(\mathfrak{M}, \iota) \in \widetilde{Z}^N_G(A)$ then we obtain an element $C_{\mathfrak{M},\iota} \in G(\mathfrak{S}_A[\frac{1}{E(u)}])$ giving the isomorphism

$$\varphi^* \mathcal{E}^0 \xrightarrow{\varphi^* \iota^{-1}} \varphi^* \mathfrak{M}[\tfrac{1}{E(u)}] \xrightarrow{\varphi_{\mathfrak{M}}} \mathfrak{M}[\tfrac{1}{E(u)}] \xrightarrow{\iota} \mathcal{E}^0$$

We say that $C_{\mathfrak{M},\iota}$ represents the Frobenius on $\mathfrak{M}$ relative to ι.

Construction 15.3. We have morphisms

$$\begin{tikzcd} & \widetilde{Z}_G \arrow[dl, "\Gamma"'] \arrow[dr, "\Psi"] & \\ Z_G & & \mathrm{Gr}_{\widetilde{G}} \end{tikzcd}$$

where Γ forgets the choice of trivialisation and

$$\Psi(\mathfrak{M}, \iota) := (\mathfrak{M}, \varphi^* \iota \circ \varphi_{\mathfrak{M}}^{-1})$$

Here $(\mathfrak{M}, \varphi^*\iota \circ \varphi_{\mathfrak{M}}^{-1})$ is viewed as a pair consisting of a $\widetilde{G}$-torsor on $\operatorname{Spec} A[[u]]$ and a trivialisation after inverting $E(u)$, and determines as an A-valued point of $\mathrm{Gr}_{\widetilde{G}}$ via Lemma 4.3; since A is p-adically complete the $E(u)$-adic completion of $A[u]$ coincides with $A[[u]]$. Concretely, if $C_{\mathfrak{M},\iota} \in G(\mathfrak{S}_A[\frac{1}{E(u)}]) = \widetilde{G}(A[[u]][\frac{1}{E(u)}])$ represents the Frobenius on $\mathfrak{M}$ relative to ι, then $\Psi(\mathfrak{M},\iota) = (\mathcal{E}^0, C_{\mathfrak{M},\iota}^{-1})$. It is easy to see that Γ and Ψ are respectively torsors for the following two actions of the group scheme[1] L^+G over $\mathcal{O}$ given by $A \mapsto G(\mathfrak{S}_A)$ on $\widetilde{Z}_G$:

$$g \cdot_\varphi (\mathfrak{M},\iota) = (\mathfrak{M}, g \circ \iota), \qquad g \cdot_{trans} (\mathfrak{M},\iota) = (\mathfrak{M}_g, \iota)$$

for $\mathfrak{M}_g \in Z_G(A)$ the Breuil–Kisin with underlying G-torsor $\mathfrak{M}$ and with Frobenius represented (in the sense of Remark 15.2) by $gC_{\mathfrak{M},\iota}$.

15.4. For an alternative viewpoint on Construction 15.3 let LG denote the group ind-scheme over $\mathcal{O}$ given by $A \mapsto G(\mathfrak{S}_A[\frac{1}{E(u)}])$. Then $(\mathfrak{M},\iota) \mapsto C_{\mathfrak{M},\iota}$ gives an isomorphism $\widetilde{Z}_G \cong LG$ (or rather the p-adic completion of LG). Under this isomorphism the $\cdot_\varphi$-action identifies with the action of L^+G via φ-conjugation $C \mapsto g^{-1}C\varphi(g)$ while the $\cdot_{trans}$-action identities with left multiplication. Therefore, the diagram in Construction 15.3 identifies with

$$\begin{tikzcd} & LG \arrow[dl] \arrow[dr, "C \mapsto C^{-1}"] & \\ \left[LG/_\varphi L^+G\right] & & \left[LG/L^+G\right] \cong \mathrm{Gr}_{\widetilde{G}} \end{tikzcd}$$

where $LG/_\varphi L^+G$ indicates the quotient by φ-conjugation and LG/L^+G indicates the quotient by right multiplication.

An issue with $\widetilde{Z}_G$ is that it is not of finite type over $\mathcal{O}$. To address this we will consider the certain quotients. These ideas go back to [PR09, 2.2]. See also [Lin23, §3.3] which does essentially the same as that done here.

Definition 15.5. For $N \geq 1$ let $U_{G,N} \subset L^+G$ denote the subgroup with A-valued points

$$\ker\left(G(\mathfrak{S}_A) \to G(\mathfrak{S}_A/u^N)\right)$$

and set $\mathcal{G}_{G,N} = L^+G/U_{G,N}$.

Proposition 15.6. *Let $X \subset \mathrm{Gr}_{\widetilde{G}}$ be a closed subscheme on which p is nilpotent. Then, for $N \geq N_0$ (with N_0 depending on X),*

$$[\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X/_\varphi U_{G,N}] \cong [\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X/U_{G,N}]$$

(the quotient on the left being by the $\cdot_\varphi$ action and that on the right action by the $\cdot_{trans}$-action). In particular, the map Ψ induces a morphism

$$\Psi_N : [\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X/_\varphi U_{G,N}] \to X$$

which is a torsor for the group scheme $\mathcal{G}_{G,N}$.

Proof. The isomorphism $[\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X/_\varphi U_{G,N}] \cong [\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X/U_{G,N}]$ follows from the concrete assertion that there exists $N \geq 1$ so that for any $C \in LG(A) = G(\mathfrak{S}_A[\frac{1}{E(u)}])$ representing an A-valued point in X one has:

- If $g_0 \in U_{G,N}(A)$ then $g_0^{-1}C\varphi(g_0) = gC$ for a unique $g \in U_{G,N}(A)$.

[1] Note that this is not the same group scheme as that defined in Notation 4.7.

- If $g \in U_{G,N}(A)$ then there exists a unique $g_0 \in U_{G,N}(A)$ for which $g_0^{-1}C\varphi(g_0) = gC$.

When $G = \mathrm{GL}_n$ this is shown in [Bar21, 9.6] (following arguments in [PR09, 2.2]). For general G one chooses a faithful representation into GL_n. The first point then follows immediately from the statement for GL_n. For the second point one recalls that, in the case of GL_n, one constructs $g_0 \in U_{\mathrm{GL}_n,N}(A) = 1 + u^N \operatorname{Mat}(\mathfrak{S}_A)$ as the limit of a u-adically converging sequence of matrices in $U_{\mathrm{GL}_n,N}(A)$. If g and C are in $G(\mathfrak{S}_A[\frac{1}{E(u)}])$ then g_0 will be the limit of a convergent sequence in $G(\mathfrak{S}_A[\frac{1}{E(u)}]) \cap U_{\mathrm{GL}_n,N}(A) = U_{G,N}(A)$. Thus $g_0 \in U_{G,N}(A)$ also, and the proposition follows. □

Corollary 15.7. *Let μ be a Hodge type and assume that for each $\kappa_0 : k \to \mathbb{F}$*

$$\sum_{i=1}^{e} \langle \alpha^\vee, \mu_i \rangle \leq p$$

for all roots $\alpha^\vee$. Then there exists a closed subfunctor $Z_{G,\mu,\mathbb{F}}$ of $Z_G \otimes_{\mathcal{O}} \mathbb{F}$ represented by an algebraic stack, of finite type over $\operatorname{Spec}\mathbb{F}$*, with the property that $\mathfrak{M} \in Z_{G,\mu,\mathbb{F}}(A)$ if and only if*

- *For any A-algebra A' and any trivialisation ι of $\mathfrak{M} \otimes_A A'$ one has $\Psi(\mathfrak{M} \otimes_A A', \iota) \in M_\mu \otimes_{\mathcal{O}} \mathbb{F}$.*

Furthermore, $\dim Z_{G,\mu,\mathbb{F}} = \sum_{\kappa : K \to E} \dim \widetilde{G}/P_{\mu_\kappa}$.

Proof. Applying Proposition 15.6 with $X = M_\mu \otimes_{\mathcal{O}} \mathbb{F}$ shows that $[\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X /_{\varphi} U_{G,N}]$ is, for large enough N, a finite type $\mathbb{F}$-scheme of dimension

$$\dim \mathcal{G}_{G,N} + \sum_{\kappa : K \to E} \dim \widetilde{G}/P_{\mu_\kappa}$$

To construct $Z_{G,\mu,\mathbb{F}}$ we descend this closed subscheme along the morphism Ψ_N. For this we need that $[\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X /_{\varphi} U_{G,N}]$ is stable under the $g \cdot_\varphi (\mathfrak{M}, \iota)$ action of $\mathcal{G}_{G,N}$. Since $C_{\mathfrak{M}, g \circ \iota} = g^{-1} C_{\mathfrak{M},\iota} \varphi(g)$ this stability is equivalent to asking that the A-valued points of each $M_\mu \otimes_{\mathcal{O}} \mathbb{F} \subset \mathrm{Gr}_{\widetilde{G}}$ are stable under the action of $\widetilde{G}(A[u^p])$.

For this notice that if $\sum_{i=1}^{e} \langle \alpha^\vee, \mu_i \rangle \leq p$ then $g \in \widetilde{G}(A[u^p])$ acts on $Y_{G,\leq\mu} \otimes_{\mathcal{O}} \mathbb{F}$ as $g_0 := g$ modulo u^p (this is clear from the definition). Since $M_\mu \otimes_{\mathcal{O}} \mathbb{F} \subset Y_{G,\leq \mu} \otimes_{\mathcal{O}} \mathbb{F}$ (see Proposition 5.9) the claim reduces to the claim that $M_\mu \otimes_{\mathcal{O}} \mathbb{F}$ is stable under the action of G, and this is immediate. □

Remark 15.8. We do not know whether Corollary 15.7 remains true with the bound $\sum_{i=1}^{e} \langle \alpha^\vee, \mu_i \rangle \leq p$ replaced with the more natural bound $\sum_{i=1}^{e} \langle \alpha^\vee, \mu_i \rangle \leq e + p - 1$.

Corollary 15.9. *Let $H \subset G$ be an embedding of reductive groups. Then the induced morphism*

$$Z_H \to Z_G$$

is representable by schemes, and of finite type.

Proof. The well-known fact that $\mathrm{Bun}_H \to \mathrm{Bun}_G$ is representable by schemes implies that, for any A-valued point of Z_G, $Z_H \times_{Z_G} \operatorname{Spec} A$ is representable by a closed subscheme of an $\mathfrak{S}_A$-scheme. To check this scheme is of finite type over A we can assume that A is a Noetherian $\mathcal{O}$-algebra A on which p is nilpotent. After replacing A by an fppf-cover we can factor $\operatorname{Spec} A \to Z_G$ through $[\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X /_{\varphi} U_{G,N}]$ for sufficient large N and some $X \subset \mathrm{Gr}_{\widetilde{G}}$. We can also assume X is actually a closed subscheme of $\mathrm{Gr}_{\widetilde{H}}$. Writing $\widetilde{Z}^N_{G,X} := [\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X /_{\varphi} U_{G,N}]$ we then have a sequence of morphisms

$$\widetilde{Z}^N_{H,X} \times_{\widetilde{Z}^N_{G,X}} \times \operatorname{Spec} A \to Z_H \times_{Z_G} \operatorname{Spec} A \to \operatorname{Spec} A$$

The composite is of finite type since the same is true of $[\widetilde{Z}_H \times_{\mathrm{Gr}_{\widetilde{H}}} X /_{\varphi} U_{H,N}] \to [\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} X /_{\varphi} U_{G,N}])$ (in fact this is a closed immersion). As A is Noetherian it follows that $Z_H \times_{Z_G} \operatorname{Spec} A$ is of finite type also. □

Remark 15.10. For an embedding $H \subset G$ the analogous morphism between moduli spaces of shtuka's in representable by schemes, and additionally finite and unramified [Bre19, Yun22]. One expects that the same is true for $Z_H \to Z_G$, and it seems that the arguments of loc. cit. will go through largely unchanged. Since we do not need this additional level of control we do not try to give any details.

16. Crystalline Breuil–Kisin modules

Here we discuss the link between Breuil–Kisin modules and crystalline representations, by extending the discussion from [Bar21, §10] from GL_n to G.

Definition 16.1. Let A be a p-adically complete $\mathcal{O}$-algebra topologically of finite type and recall the $\mathfrak{S}_A$-algebra $A_{\mathrm{inf},A}$ which is equipped with a Frobenius extending that on $\mathfrak{S}_A$ and a continuous action of G_K commuting with the Frobenius. By an action of G_K-action on $\mathfrak{M} \in Z_G(A)$ we mean a collection of morphisms in $Z_G(A)$

$$x_\sigma : \mathfrak{M} \otimes_{\mathfrak{S}_A,\sigma} A_{\mathrm{inf},A} \xrightarrow{\sim} \mathfrak{M} \otimes_{\mathfrak{S}_A} A_{\mathrm{inf},A}, \qquad \sigma \in G_K$$

satisfying $x_{\sigma\tau} = x_\sigma \circ \sigma^* x_\tau$ and $x_1 = \mathrm{id}$. Such a G_K-action is crystalline if, for each representation $\chi : G \to \mathrm{GL}_n$ over $\mathcal{O}$, the induced G_K-action satisfies (recall the element $\mu = [\epsilon] - 1 \in A_{\mathrm{inf},A}$)

$$\sigma(m) - m \in \mathfrak{M}(\rho)^\chi \otimes_{\mathfrak{S}_A} u\varphi^{-1}(\mu) A_{\mathrm{inf},A}, \qquad \sigma_\infty(m) - m = 0$$

for all $m \in \mathfrak{M}^\chi$ and all $\sigma \in G_K, \sigma_\infty \in G_{K_\infty}$. Write $Y_G(A)$ for the groupoid consisting of $\mathfrak{M} \in Z_G(A)$ equipped with a crystalline G_K-action.

We say that a continuous representation $\rho : G_K \to G(A)$ is crystalline if $\chi \circ \rho$ is crystalline in the sense of [Fon94b] for every representation χ of G (equivalent for a single faithful χ).

Theorem 16.2. *Let A be a finite flat $\mathcal{O}$-algebra. Then to each $(\mathfrak{M}, x) \in Y_G(A)$ there exists is a uniquely determined (up to isomorphism) crystalline representation $\rho : G_K \to G(A)$ together with a φ, G_K-equivariant identification*

$$\mathfrak{M} \otimes_{\mathfrak{S}_A} W(C^\flat)_A \cong \rho \otimes_A W(C^\flat)_A$$

of G-torsors. If A is a discrete valuation ring then every crystalline ρ arises in this way from a unique (up to isomorphism) such $(\mathfrak{M}, x)$.

Proof. The first part follows immediately from the assertion for GL_n (see [Bar20, 2.1.12]). The second part does not immediately follow from the case of GL_n because the construction in [Kis06] of $\mathfrak{M}$ from any $\rho : G_K \to \mathrm{GL}_n(A)$, while functorial and tensor compatible, is not exact. In particular, the construction cannot, a priori, be used to associate the G-torsor $\mathfrak{M}$ to $\rho : G_K \to G(A)$.

Fortunately, this issue can easily be addressed because Kisin's construction actually produces an exact tensor functor sending a crystalline representations

$\rho : G_K \to \mathrm{GL}_n$ onto a vector bundle $\mathfrak{M}(\rho)^*$ on the $D^* = \operatorname{Spec} \mathfrak{S}_{\mathcal{O}} \smallsetminus \{u = p = 0\}$ equipped with a Frobenius isomorphism after inverting $E(u)$. As explained in e.g. [Lev15, 2.3.6] such an $\mathfrak{M}(\rho)^*$ can be interpreted as a pair of projective φ-modules respectively over $\mathfrak{S}_{\mathcal{O}}[\frac{1}{p}]$ and the p-adic completion $\mathcal{O}_{\mathcal{E},A}$ of $\mathfrak{S}_A[\frac{1}{u}]$, together with a comparison isomorphism over $\mathcal{O}_{\mathcal{E},A}[\frac{1}{p}]$. The former is constructed in [Kis06, 1.3.15] (and we look into this construction in more detail in Section 17) and the latter is the etale φ-module associated to ρ as in e.g. [Kis06, §2.1]. Then $\mathfrak{M}(\rho)$ is obtained using the equivalence between vector bundles on D^* and $\operatorname{Spec} \mathfrak{S}_{\mathcal{O}}$ [Ans22, 1.2] (this extension is where exactness is lost).

Since $\rho \mapsto \mathfrak{M}(\rho)^*$ is exact and tensor compatible applying the construction to a crystalline representation valued in $G(\mathcal{O})$ produces a G-torsor on D^* equipped with a Frobenius after inverting $E(u)$. Then [Ans22, 1.2] can be applied again to extend this G-torsor to a G-Breuil–Kisin module, producing the $\mathfrak{M}$ associated to ρ. $\square$

We say that a crystalline representation $\rho : G_K \to G(A)$ has Hodge type μ if $\chi \circ \rho$ has Hodge type $\chi \circ \mu$ for any representation $\chi : G \to \mathrm{GL}_n$ of G.

Proposition 16.3. *For each Hodge type μ there exists a closed subfunctor Y_G^μ of Y_G which is represented by an $\mathcal{O}$-flat p-adic algebraic formal stack (in the sense* [EG23, A7]*) Y_G^μ of topologically finite type over $\mathcal{O}$ and is uniquely determined by the property that its groupoid of A-valued points, for any finite flat $\mathcal{O}$-algebra A, is canonically equivalent to the full subcategory $Y_G^\mu(A)$ of $Y_G(A)$ consisting of those $\mathfrak{M}$ for which the associated crystalline representation ρ as in Theorem 16.2 has Hodge type μ.*

Proof. When $G = \mathrm{GL}_n$ this is [Bar21, 10.7]. For general G one chooses a faithful representation $\chi : G \to \mathrm{GL}_n$. Corollary 15.9 shows that the projection

$$Y^{\chi\circ\mu}_{\mathrm{GL}_n} \times_{Z_{\mathrm{GL}_n}} Z_G \to Y^{\chi\circ\mu}_{\mathrm{GL}_n}$$

is representable by finite type schemes and so $Y^{\chi\circ\mu}_{\mathrm{GL}_n} \times_{Z_{\mathrm{GL}_n}} Z_G$ is a p-adic algebraic formal stack of topologically finite type over $\mathcal{O}$. One takes Y_G^μ as its $\mathcal{O}$-flat closure of $Y^{\chi\circ\mu}_{\mathrm{GL}_n} \times_{Z_{\mathrm{GL}_n}} Z_G$ (in the sense of [EG23, p.230]). $\square$

We finish this section by explaining the relationship between Y_G^μ and G-crystalline deformation rings. Fix a continuous homomorphism $\overline{\rho} : G_K \to G(\mathbb{F})$ and let $R^\square_{\overline{\rho}}$ denote the corresponding framed deformation ring over $\mathcal{O}$, i.e. the unique complete local Noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}$ equipped with a continuous homomorphism $\rho^{\mathrm{univ}} : G_K \to G(R^\square_{\overline{\rho}})$ satisfying $\rho^{\mathrm{univ}} \otimes_{R^\square_{\overline{\rho}}} \mathbb{F} = \overline{\rho}$ and universal amongst all such rings with this property.

Theorem 16.4. *For each Hodge type μ there exists a unique $\mathcal{O}$-flat reduced quotient $R^{\square,\mathrm{cr},\mu}_{\overline{\rho}}$ of $R^\square_{\overline{\rho}}$ with the property that a homomorphism $R^\square_{\overline{\rho}} \to A$ with A a finite flat $\mathcal{O}$-algebra factors through $R^{\square,\mathrm{cr},\mu}_{\overline{\rho}}$ if and only if the composite*

$$G_K \xrightarrow{\rho^{\mathrm{univ}}} G(R^\square_{\overline{\rho}}) \to G(A)$$

is crystalline of Hodge type μ. Furthermore,

$$\dim_{\mathcal{O}} R^{\square,\mathrm{cr},\mu}_{\overline{\rho}} = \dim_{\mathcal{O}} G + \sum_{i=1}^{e} \dim \widehat{G}/P_{\mu_i}$$

Proof. This is a special case of the main result of [Kis08] when $G = \mathrm{GL}_n$ and of [Bal12] for general G. See also [BG19, Theorem A]. □

Construction 16.5. Let A be any complete local Noetherian $\mathcal{O}$-algebra with finite residue field and let $\rho : G_K \to G(A)$ be a continuous representation. Consider the functor which sends any p-adically complete A-algebra $\mathcal{O}$ which is topologically of finite type over $\mathcal{O}$ on the set tuples $(\mathfrak{M}, x, \alpha, \beta)$ for which $(\mathfrak{M}, x) \in Y_G^\mu(A')$, $\alpha : A \to B$ is a continuous homomorphism, and β is a φ, G_K-equivariant identification

$$\mathfrak{M} \otimes_{\mathfrak{S}_{A'}} W(C^\flat)_{A'} \cong \rho \otimes_A W(C^\flat)_{A'}$$

After choosing a faithful representation it follows from e.g. [EG23, 4.5.26] that this functor is represented by the $\mathfrak{m}_A$-adic completion of a projective A-scheme whose $\mathcal{O}$-flat closure we denote by $\mathcal{L}_\rho^\mu$. Then $\mathcal{L}_\rho^\mu$ has the property that the structure morphism $\mathcal{L}_\rho^\mu \to \operatorname{Spec} A$ becomes a closed immersion after inverting p. The scheme theoretic image of this morphism corresponds to a quotient $A^{\mathrm{cr},\mu}$ of A with the property that a homomorphism $A \to B$ into a finite flat $\mathcal{O}$-algebra B factors through $\mathcal{L}_{\overline{\rho}}^\mu$ if and only if $G_K \xrightarrow{\rho} G(A) \to G(B)$ is crystalline of Hodge type μ.

For a given continuous $\overline{\rho} : G_K \to G(\mathbb{F})$ set $\mathcal{L}_{\overline{\rho}}^{\mathrm{cr},\mu} = \mathcal{L}_{\rho^{\mathrm{univ}}}$ for $\rho^{\mathrm{univ}} : G_K \to G(R_{\overline{\rho}}^\square)$ the universal deformation of ρ. Then Construction 16.5 shows that $R_{\overline{\rho}}^{\square,\mathrm{cr},\mu}$ is the scheme theoretic image of $\mathcal{L}_{\overline{\rho}}^{\mathrm{cr},\mu}$.

Lemma 16.6. *For any continuous homomorphism $\overline{\rho} : G_K \to G(\mathbb{F})$ there is a formally smooth morphism*

$$\mathcal{L}_{\overline{\rho}}^{\mathrm{cr},\mu} \otimes_{\mathcal{O}} \mathbb{F} \to Y_G^\mu \otimes_{\mathcal{O}} \mathbb{F}$$

of relative dimension $\dim_{\mathcal{O}} G$ which induces, for any p-adically complete $\mathcal{O}$-algebra of topologically finite type over $\mathcal{O}$, the functor $(\mathfrak{M}, x, \alpha, \beta) \mapsto (\mathfrak{M}, x)$.

Proof. The lifting property describing formal smoothness can be checked on p-adically complete $\mathcal{O}$-algebras factoring through $R_{\overline{\rho}}^{\square,\mathrm{cr},\mu}$, and so we can assume the ring is a complete local Noetherian ring with finite residue field. The lifting can therefore be deduced from the main result of [Dee01]. This lifting is unique up to G-conjugation which shows that the relative dimension is as claimed. □

Corollary 16.7.
1. *Let A be an Artin local $\mathcal{O}$-algebra and $(\mathfrak{M}, x) \in Y_G^\mu(A)$. Then there exists a finite flat $\mathcal{O}$-algebra A°, with a morphism $A^\circ \to A$, and $(\mathfrak{M}^\circ, x^\circ) \in Y_G^\mu(A^\circ)$ whose image under $Y_G^\mu(A^\circ) \to Y_G^\mu(A)$ is $(\mathfrak{M}, x)$.*
2. *$Y_G^\mu \otimes_{\mathcal{O}} \mathbb{F}$ has dimension $\sum_{i=1}^e \dim \widetilde{G}/P_{\mu_i}$.*

Proof. Using Lemma 16.6 this follows from analogous assertions for $\mathcal{L}_\rho^\mu$, of which (1) is a consequence of [Bar20, 4.1.2] and (2) is a consequence of the dimension formula for $R_{\overline{\rho}}^{\square,\mathrm{cr},\mu}$. □

17. The shape of Frobenius

For the results of this section it is necessary to assume that the compatible system π^{1/p^∞} is chosen so that $K_\infty \cap K(\mu_{p^\infty}) = K$ whenever μ^{p^∞} is a compatible system of primitive p-th power roots of unity. When $p > 2$ this is automatic and, while not automatic when $p = 2$, it follows from [Wan22] that π can be chosen so that this is the case.

Theorem 17.1. *Assume that μ is a Hodge type satisfying*

$$\sum_{i=1}^{e}\langle\alpha^\vee,\mu_i\rangle \leq \frac{p-1}{\nu}+1, \qquad \nu = \max_{i\neq j}\{v(\pi_i-\pi_j)\}$$

where v denotes the valuation on $\mathcal{O}$ with $v(\pi_i)=1$ for one (equivalently all) i. Then the morphism $Y_G^\mu \otimes_{\mathcal{O}} \mathbb{F} \to Z_G$ which forgets the crystalline G_K-action factors through $Z_{G,\mu,\mathbb{F}}$.

Remark 17.2.

(1) If p does not divide e, i.e. if K is tamely ramified over $\mathbb{Q}_p$, then $\nu = 1$. To see this recall $E(u) = \prod_{i=1}^{e}(u-\pi_i)$ and so

$$\frac{d}{du}E(u) = \sum_{i=1}^{e}\prod_{j\neq i}(u-\pi_j)$$

Therefore $\frac{d}{du}E(u)|_{u=\pi_i} = \prod_{j\neq i}(\pi_i-\pi_j)$ has valuation $\sum_{j\neq i}v(\pi_j-\pi_i)$. On the other hand, since $E(u)\equiv u^e$ modulo p we have

$$\frac{d}{du}E(u)|_{u=\pi_i} = e\pi_i^{e-1} \text{ modulo } p$$

and so, if e is prime to p, then $e-1 = \sum_{j\neq i}v(\pi_j-\pi_i)$. As $v(\pi_j-\pi_i)\geq 1$ we must have each $v(\pi_j-\pi_i)=1$.

(2) If each μ_i is strictly dominant then $\sum_{i=1}^{e}\langle\alpha^\vee,\mu_i\rangle \geq e$ and so for the bound in Theorem 17.3 to hold we must have

$$e \leq \frac{p-1}{\nu}+1$$

In particular, (1) implies $\nu=1$. Therefore, the bound in Theorem 17.3 is equivalent to asking that

$$\sum_{i=1}^{e}\langle\alpha^\vee,\mu_i\rangle \leq p$$

for each root $\alpha^\vee$.

In order to prove Theorem 17.1 it suffices to show such a factorisation on the level of Artin local $\mathbb{F}$-algebras (see for example [Bar21, 15.2]). Using the lifting result of Corollary 16.7 we therefore reduce Theorem 17.1 to the following:

Theorem 17.3. *Let A be a finite flat $\mathcal{O}$-algebra and suppose that $(\mathfrak{M}(\rho),x)\in Y_G^\mu(A)$ corresponds as in Theorem 16.2 to a crystalline representation ρ of Hodge type μ. Assume that, for all roots $\alpha^\vee$ of $\widetilde{G}$,*

$$\sum_{i=1}^{e}\langle\alpha^\vee,\mu_i\rangle \leq \frac{p-1}{\nu}+1, \qquad \nu = \max_{i\neq j}\{v(\pi_i-\pi_j)\}$$

where v denotes the valuation on $\mathcal{O}$ with $v(\pi_i)=1$ for one (equivalently all) i. Then

$$\Psi(\mathfrak{M}(\rho),\iota)\otimes_{\mathcal{O}}\mathbb{F} \in M_\mu(A\otimes_{\mathcal{O}}\mathbb{F})$$

for any trivialisation ι of $\mathfrak{M}(\rho)$

The rest of this section will be devoted to the proof of the theorem. The first step is to realise the Hodge type μ in terms of $\mathfrak{M}(\rho)$. We will see that this is easy after inverting p.

17.4. We begin by introducing some power series rings in which p had been inverted:

- Let $\widehat{\mathfrak{S}}_A$ denote the $E(u)$-adic completion of $\mathfrak{S}_A[\frac{1}{p}]$. Notice that since $E(u)$ generates the kernel of the surjection $\widehat{\mathfrak{S}}_A \to K \otimes_{\mathbb{Z}_p} A$ sending $u \mapsto \pi$ this surjection has a unique splitting, via which we view $\widehat{\mathfrak{S}}_A$ as a $K \otimes_{\mathbb{Z}_p} A$-module.
- Let $\widehat{\mathfrak{S}}_{A,i}$ denote the $(u-\pi_i)$-adic completion of $\mathfrak{S}_A[\frac{1}{p}]$ and identify this ring with $(K_0 \otimes_{\mathbb{Z}_p} A)[[u-\pi_i]]$. As in 4.4 we have an isomorphism $\widehat{\mathfrak{S}}_A \cong \prod_{i=1}^e \widehat{\mathfrak{S}}_{A,i}$ and this allows us to consider the Taylor expansion around $u = \pi_i$ of any $f \in \widehat{\mathfrak{S}}_A$: it is the power series
$$\sum_{n\geq 0} f_n(u-\pi_i)^n, \qquad f_n \in K_0 \otimes_{\mathbb{Z}_p} A$$
corresponding to the image of f in $\widehat{\mathfrak{S}}_{A,i}$.
- Let $\mathcal{O}^{\mathrm{rig}}$ denote the subring of $K_0[[u]]$ consisting of power series convergent on the open unit disk and set $\mathcal{O}^{\mathrm{rig}}_A = A \otimes_{\mathbb{Z}_p} \mathcal{O}^{\mathrm{rig}}$ whenever A is a finite $\mathcal{O}$-algebra. We set $\lambda = \prod_{i=0}^\infty \frac{\varphi^n(E(u))}{E(0)} \in \mathcal{O}^{\mathrm{rig}}$ and we view $\mathcal{O}^{\mathrm{rig}}_A[\frac{1}{\varphi(\lambda)}]$ as an $\widehat{\mathfrak{S}}_A$-algebra by sending an element onto its Taylor series around $u = \pi$. We write φ for the unique extension of φ on $\mathfrak{S}_A$ to $\mathcal{O}^{\mathrm{rig}}_A$.

Notice that the composite $\mathcal{O}^{\mathrm{rig}}_A[\frac{1}{\varphi(\lambda)}] \to \widehat{\mathfrak{S}}_{A,i}$ is injective for each i and so we frequently abuse notation by writing
$$f = \sum_{n\geq 0} f_n(u-\pi_i)^n$$
whenever $f \in \mathcal{O}^{\mathrm{rig}}_A[\frac{1}{\varphi(\lambda)}]$.

17.5. Next we recall some aspects of the construction of $\mathfrak{M}(\rho)[\frac{1}{p}]$ from [Kis06] when $G = \mathrm{GL}_n$. First, the filtered φ-module $D(\rho)$ associated to $\rho[\frac{1}{p}]$ is used to construct a projective $\mathcal{O}^{\mathrm{rig}}_A = \mathcal{O}^{\mathrm{rig}} \otimes_{\mathbb{Z}_p} A$-module $\mathcal{M}(\rho)$ together with an isomorphism
$$\varphi^*\mathcal{M}(\rho)[\tfrac{1}{\lambda}] \xrightarrow{\sim} \mathcal{M}(\rho)[\tfrac{1}{\lambda}]$$
See [Kis06, §1.2]. There are two key consequences of this construction:

- There exists a φ-equivariant isomorphism
$$\xi : \varphi^*\mathcal{M}(\rho)[\tfrac{1}{\varphi(\lambda)}] \cong D(\rho) \otimes_{K_0 \otimes_{\mathbb{Z}_p} A} \mathcal{O}^{\mathrm{rig}}_A[\tfrac{1}{\varphi(\lambda)}]$$
See [Kis06, 1.2.6].
- After extending scalars to $\widehat{\mathfrak{S}}_A$ we obtain isomorphisms $\varphi^*\mathcal{M}(\rho) \otimes_{\mathcal{O}^{\mathrm{rig}}_A} \widehat{\mathfrak{S}}_A \cong D(\rho) \otimes_{K_0 \otimes_{\mathbb{Z}_p} A} \widehat{\mathfrak{S}}_A \cong D(\rho)_K \otimes_{K \otimes_{\mathbb{Z}_p} A} \widehat{\mathfrak{S}}_A$ under which
$$\mathcal{M}(\rho) \otimes_{\mathcal{O}^{\mathrm{rig}}_A} \widehat{\mathfrak{S}}_A = \sum_{i\in\mathbb{Z}} \mathrm{Fil}^i(D(\rho)_K) \otimes_{K \otimes_{\mathbb{Z}_p} A} E(u)^{-i}\widehat{\mathfrak{S}}_A \tag{17.6}$$
See [Kis06, 1.2.1].

Using that $D(\rho)$ comes from a crystalline representation (i.e. is an admissible filtered φ-module) Kisin then shows [Kis06, 1.3.8] that $\mathcal{M}(\rho)$ descends uniquely to the projective $\mathfrak{S}_A[\frac{1}{p}]$-module $\mathfrak{M}(\rho)[\frac{1}{p}]$.

All the constructions from 17.5 are functorial in ρ, and compatible with exact sequences and tensor products. Therefore, the Tannakian formalism ensures that the observations in 17.5 remain valid (after interpreting (17.6) as in 4.10) when ρ is valued in a general G. As a consequence we deduce:

Corollary 17.7. *For any trivialisation β of $D(\rho)$ over* $\operatorname{Spec} K_0 \otimes_{\mathbb{Z}_p} A$*, the pair*

$$(\mathfrak{M}(\rho) \otimes_{\mathfrak{S}_A} \widehat{\mathfrak{S}}_A, \beta \circ \xi \circ \varphi_{\mathfrak{M}}^{-1})$$

consisting of a G-torsor on $\operatorname{Spec} \widehat{\mathfrak{S}}_A$ *and a trivialisation after inverting $E(u)$, defines an $A[\frac{1}{p}]$-valued point of $M_\mu \subset \operatorname{Gr}_{\widetilde{G}}$. Equivalently, if $X_{\xi,\beta}$ denotes the automorphism*

$$\mathcal{E}^0 \xrightarrow{\varphi^*\iota^{-1}} \varphi^*\mathfrak{M}(\rho) \otimes_{\mathfrak{S}_A} \mathcal{O}_A^{\mathrm{rig}}[\tfrac{1}{\varphi(\lambda)}] \xrightarrow{\xi} D(\rho) \otimes_{K_0 \otimes_{\mathbb{Z}_p} A} \mathcal{O}_A^{\mathrm{rig}}[\tfrac{1}{\varphi(\lambda)}] \xrightarrow{\beta} \mathcal{E}^0$$

of the trivial G-torsor then

$$X_{\xi,\beta} \cdot \Psi(\mathfrak{M}(\rho), \iota)[\tfrac{1}{p}] \in M_\mu(A[\tfrac{1}{p}])$$

This requires no bound on the Hodge type μ.

In order to use Corollary 17.7 to prove Theorem 17.3 we need to control the denominators appearing in $X_{\xi,\beta}$. Following ideas of [GLS14] we do this by first deriving some kind of intergrality of a differential operator associated to $X_{\xi,\beta}$. Set $S_{\max} = W(k)[[u, \frac{u^e}{p}]] \cap \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ and $S_{\max,A} = S_{\max} \otimes_{\mathbb{Z}_p} A$ for any finite $\mathcal{O}$-algebra A.

Proposition 17.8. *Assume $G = \mathrm{GL}_n$ and define a differential operator N_∇ on $\varphi^*\mathcal{M}(\rho)[\frac{1}{\varphi(\lambda)}] \cong D(\rho) \otimes_{K_0 \otimes_{\mathbb{Z}_p} A} \mathcal{O}_A^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$ over $\partial := u\frac{d}{du}$ by setting $N_\nabla(d) = 0$ for $d \in D(\rho)$. The assumption that $K_\infty \cap K(\mu_{p^\infty}) = K$ ensures the matrix of N_∇ relative to the trivialisation $\varphi^*\iota$ of $\varphi^*\mathcal{M}(\rho)[\frac{1}{\varphi(\lambda)}]$ has entries in*

$$u^p \varphi(S_{\max,A})$$

Again this requires no bound on the Hodge type μ.

Proof. The proof will be given in Section 20 below. The essential idea is to relate N_∇ and the G_K-action on $\mathfrak{M} \otimes_{\mathfrak{S}_A} A_{\mathrm{inf},A}$ after basechanging to an appropriate period ring, and exploit the integrality of the G_K-action. □

Remark 17.9. What is proved in [GLS14, 4.7] (when $p > 2$) and [Wan22, 4.1] (when $p = 2$ and π is chosen so that $K_\infty \cap K(\mu^{p^\infty}) = K$) is that the entries of the matrix representing N_∇ are contained in

$$u^p \left(W(k)[[u^p, \tfrac{u^{ep}}{p}]][\tfrac{1}{p}] \cap S \right) \otimes_{\mathbb{Z}_p} A \tag{17.10}$$

where S denotes the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. This is slightly weaker than Proposition 17.8 (though for the purposes of this paper it makes no difference because the calculations in the first paragraph of Corollary 17.11 below also go through using (17.10), see [GLS15, 2.3.9]). We have stated the stronger result here because it may be useful when considering Hodge types beyond the bounds imposed in this paper.

Corollary 17.11. *Continue to assume $G = \mathrm{GL}_n$ and fix a trivialisation β of $D(\rho)$ over* $\operatorname{Spec} K_0 \otimes_{\mathbb{Z}_p} A$*. We then view the automorphism $X_{\xi,\beta}$ from Corollary 17.7 as a matrix and, as in 17.4, write its Taylor expansion around $u = \pi_i$ as*

$$X_{\xi,\beta} = \sum_{n \geq 0} X_{i,n}(u - \pi_i)^n$$

Then

$$X_{i,0}^{-1} X_{i,n} \in \mathrm{Mat}(\pi_i^{p-n} W(k) \otimes_{\mathbb{Z}_p} A)$$

for $1 \leq n \leq p - 1$.

Proof. Let $\underline{e} = (e_1, \dots, e_n)$ denote the standard basis of $\mathcal{E}^0$. Then $\xi(\varphi^*\iota^{-1}(\underline{e})) = \beta^{-1}(\underline{e})X_{\xi,\beta}$. We can also write

$$N_\nabla(\varphi^*\iota^{-1}(\underline{e})) = \varphi^*\iota^{-1}(\underline{e})N$$

and Proposition 17.8 ensures that the matrix N has entries in $u^p\varphi(S_{\max,A})$. Therefore,

$$N = \sum_{n\geq 1} \frac{N'_n}{\pi_i^{n-1}} u^{pn}$$

for matrices N'_m with entries in $W(k)\otimes_{\mathbb{Z}_p} A$. If the Taylor expansion of N around $u = \pi_i$ is $\sum_{m\geq 0} N_m(u-\pi_i)^m$ then

$$N_m = \frac{1}{m!}\left(\frac{d}{du}\right)^m (N)|_{u=\pi_i} = \sum_{n\geq 1}\binom{pn}{m} N'_n \pi_i^{(p-1)n-m+1}$$

for $m \geq 0$. Therefore $N_0 \in \pi_i^p \operatorname{Mat}(W(k)\otimes_{\mathbb{Z}_p} A)$ and $N_m \in \pi_i^{p+e-m}\operatorname{Mat}(W(k)\otimes_{\mathbb{Z}_p} A)$ for $m = 1, \dots, p-1$.

By definition we have $N_\nabla(\xi^{-1}\circ\beta^{-1}(\underline{e})) = 0$ and so, recalling that $\partial = u\frac{d}{du}$, we have

$$\begin{aligned}\varphi^*\iota^{-1}(\underline{e})N &= N_\nabla(\varphi^*\iota^{-1}(\underline{e}))\\ &= N_\nabla(\xi^{-1}\circ\beta^{-1}(\underline{e})X_{\xi,\beta})\\ &= \xi^{-1}\circ\beta^{-1}(\underline{e})\partial(X_{\xi,\beta})\\ &= \varphi^*\iota^{-1}(\underline{e})X_{\xi,\beta}^{-1}\partial(X_{\xi,\beta})\end{aligned}$$

In other words, $\partial(X_{\xi,\beta}) = X_{\xi,\beta}N$. In terms of Taylor expansions around $u = \pi_i$ this gives the recurrence $nX_{i,n} + \pi_i(n+1)X_{i,n+1} = \sum_{j=0}^n X_{i,n-j}N_j$. Multiplying on the left by $X_{i,0}^{-1}$ gives

$$nX_{i,0}^{-1}X_{i,n} + \pi_i(n+1)X_{i,0}^{-1}X_{i,n+1} = \sum_{j=0}^n X_{i,0}^{-1}X_{i,n-j}N_j$$

for all $n \geq 0$. The corollary then follows by an easy induction, using the divisibility of the N_j from the first paragraph. □

Proof of Theorem 17.3 when $G = \mathrm{GL}_n$. Recall that if β is a trivialisation of $D(\rho)$ then

$$X_{\xi,\beta}\Psi(\mathfrak{M}(\rho),\iota)[\tfrac{1}{p}] \in M_\mu(A[\tfrac{1}{p}])$$

As in Corollary 17.11 we view $X_{\xi,\beta}$ as a matrix and, for each $1 \leq i \leq e$, we have Taylor expansions

$$X_{\xi,\beta} = \sum_{n\geq 0} X_{i,n}(u-\pi_i)^n, \qquad X_{i,n} \in \operatorname{Mat}(K_0 \otimes_{\mathbb{Z}_p} A)$$

around $u = \pi_i$. Let $g \in G(\widehat{\mathfrak{S}}_A) \cong \prod_{i=1}^e G(\widehat{\mathfrak{S}}_{A,i})$ be such that its i-th component equals $X_{i,0} \in G(K_0\otimes_{\mathbb{Z}_p} A)$. Under the isomorphism $\mathrm{Gr}_{\widetilde{G}}[\frac{1}{p}] \cong \mathrm{Gr}_{\widetilde{G},1}[\frac{1}{p}] \times \dots \times \mathrm{Gr}_{\widetilde{G},e}[\frac{1}{p}]$ the element g acts on the i-th factor by multiplication by $X_{i,0}$ and so g stabilises $M_\mu[\frac{1}{p}]$. Thus,

$$\widetilde{X}\Psi(\mathfrak{M}(\rho),\iota)[\tfrac{1}{p}] \in M_\mu(A[\tfrac{1}{p}])$$

for $\widetilde{X} = g^{-1}X_{\xi,\beta}$. For $1 \leq i \leq e$ the Taylor expansion of $\widetilde{X}$ around $u = \pi_i$ is

$$\sum_{n\geq 0}\widetilde{X}_{i,n}(u-\pi_i)^n, \qquad \widetilde{X}_{i,n} = X_{i,0}^{-1}X_{i,n}$$

and so Corollary 17.11 ensures $\widetilde{X}_{i,n} \in \pi^{p-n}\operatorname{Mat}(W(k)\otimes_{\mathbb{Z}_p} A)$ for $1 \leq n \leq p-1$. Applying Lemma 17.12 below with

$$n_i = \max_{\alpha^\vee}\{\langle \alpha^\vee, \mu_i\rangle\}$$

(the maximum taken over all roots $\alpha^\vee$ of $\widetilde{G}$) shows that the image of $\widetilde{X}$ modulo $\prod_{i=1}^e (u-\pi_i)^{n_i}$ is represented by a matrix $\widetilde{X}_{\mathrm{trunc}} \in G(\mathfrak{S}_A)$ which equals the identity in $G(\mathfrak{S}_A \otimes_{\mathcal{O}} \mathbb{F})$. This means we can write

$$\widetilde{X} = \widetilde{X}_{\mathrm{trunc}}\widetilde{X}_{\mathrm{err}}$$

with $\widetilde{X}_{\mathrm{err}} \equiv 1$ modulo $\prod_{i=1}^e (u-\pi_i)^{n_i}$. Clearly $\widetilde{X}_{\mathrm{err}}$ acts trivially on $M_\mu[\frac{1}{p}]$ and so

$$\widetilde{X}_{\mathrm{trunc}}\Psi(\mathfrak{M}(\rho),\iota)[\tfrac{1}{p}] \in M_\mu(A[\tfrac{1}{p}])$$

Since A is p-torsion free it follows that $\widetilde{X}_{\mathrm{trunc}}\Psi(\mathfrak{M}(\rho),\iota) \in M_\mu(A)$ and, since $\widetilde{X}_{\mathrm{trunc}}$ equals 1 in $G(\mathfrak{S}_A \otimes_{\mathcal{O}} \mathbb{F})$ we conclude that

$$\widetilde{X}_{\mathrm{trunc}}\Psi(\mathfrak{M}(\rho),\iota)\otimes_{\mathcal{O}}\mathbb{F} = \Psi(\mathfrak{M}(\rho),\iota)\otimes_{\mathcal{O}}\mathbb{F} \in M_\mu(A\otimes_{\mathcal{O}}\mathbb{F})$$

which finishes the proof. □

Proof of Theorem 17.3 for general G. For general G the proof is essentially identical. As when $G = \mathrm{GL}_n$ we can find $g \in G(\widehat{\mathfrak{S}}_A)$ which stabilises $M_\mu[\frac{1}{p}]$ and so that

$$\widetilde{X} := gX_{\xi,\beta} \in \ker\left(G(\widehat{\mathfrak{S}}_A) \to G(\widehat{\mathfrak{S}}_A/(u-\pi_i))\right)$$

If, as above, we set $n_i = \max_{\alpha^\vee}\{\langle\alpha^\vee,\mu_i\rangle\}$ then the action of $\widetilde{X}$ on $M_\mu[\frac{1}{p}]$ factors through its image $\widetilde{X}_{\mathrm{trunc}}$ under

$$G(\widehat{\mathfrak{S}}_A) \to G\Big(\frac{\widehat{\mathfrak{S}}_A}{\prod_{i=1}^e (u-\pi_i)^{n_i}}\Big)$$

Therefore, it suffices to show that $\widetilde{X}_{\mathrm{trunc}}$ is the image under $G(\mathfrak{S}_A) \to G(\widehat{\mathfrak{S}}_A)$ of an element in

$$\ker(G(\mathfrak{S}_A) \to G(\mathfrak{S}_A \otimes_{\mathcal{O}} \mathbb{F}))$$

and this follows from the arguments when $G = \mathrm{GL}_n$ after choosing a faithful representation of G. □

Lemma 17.12. *Let A be a finite flat $\mathcal{O}$-algebra and suppose $n_i \geq 0$ are such that*

$$\sum_{i=1}^e n_i \leq \frac{p-1}{\nu} + 1$$

for ν as in Theorem 17.3. Suppose that

$$\frac{\widehat{\mathfrak{S}}_A}{\prod_{i=1}^e (u-\pi_i)^{n_i}} \cong \prod_{i=1}^e \frac{\widehat{\mathfrak{S}}_{A,i}}{(u-\pi_i)^{n_i}}$$

maps an element f onto $(f_i)_i$ with $f_i = \sum_{n=0}^{n_i-1} f_{i,n}(u-\pi_i)^n$ for $f_{i,n} \in \pi_i^{p-n}W(k)\otimes_{\mathbb{Z}_p} A$. Then f is represented by an element in $\pi_i\mathfrak{S}_A$.

Proof. By linearity we can fix $1 \leq i \leq e$ and assume $f_j = 0$ for $i \neq j$. We need to describe the inverse of the above isomorphism and so express f in terms of f_i. For this we use the formal identity $\frac{1}{(1-y)^n} = \sum_{m\geq 0}\binom{n+m-1}{n-1}y^m$. Setting $y = \frac{u-\pi_i}{\pi_i-\pi_j}$ shows that

$$(u-\pi_j)^{n_j}\sum_{m=0}^{n_i-1}\binom{n_j+m-1}{n_j-1}\frac{(u-\pi_i)^m}{(\pi_i-\pi_j)^{m+n_j}} \equiv 1 \text{ modulo } (u-\pi_i)^{n_i}$$

Therefore f is represented by

$$F := f_i\prod_{j\neq i}(u-\pi_j)^{n_j}\sum_{m=0}^{n_i-1}\binom{n_j+m-1}{n_j-1}\frac{(u-\pi_i)^m}{(\pi_i-\pi_j)^{m+n_j}} \text{ modulo } (u-\pi_i)^{n_i}$$

(indeed,$F \equiv f_i$ modulo $(u-\pi_i)^{n_i}$ and $F \equiv 0$ modulo $(u-\pi_j)^{n_j}$ for $j \neq i$). By hypothesis the coefficient of $(u-\pi_i)^n$ in f_i has coefficient with π_i-adic valuation $\geq p-n \geq p-\nu n$ (since $\nu \geq 1$). On the other hand, the coefficient of $(u-\pi_i)^n$ in $\sum_{m=0}^{n_i-1}\binom{n_j+m-1}{n_j-1}\frac{(u-\pi_i)^m}{(\pi_i-\pi_j)^{m+n_j}}$ has π_i-adic valuation $\geq -(n+n_j)\nu$ whenever $i \neq j$. Therefore, the coefficient of $(u-\pi_i)^n$ in F has π_i-adic valuation

$$\geq p-\nu(n+\sum_{i\neq j}n_j)$$

and so we will be done if $p-\nu(n+\sum_{i\neq j}n_j) \geq 1$ for all $n = 0,\ldots,n_i-1$. In other words, if $p-\nu(-1+\sum_{j=1}^e n_j) \geq 1$ or equivalently

$$\sum_{j=1}^e n_j \leq \frac{p-1}{\nu}+1$$

which finishes the proof. □

18. Constructing Galois actions

In this section we equip $\mathfrak{M} \in Z_{G,\mu,\mathbb{F}}$ with a canonical (and unique) crystalline G_K-action, under a bound on μ. The necessary bound is very slightly stronger than asking that $\sum_{i=1}^e\langle\alpha^\vee,\mu_i\rangle \leq p+e-1$. In order to formulate it recall that if a Hodge type μ corresponds to an e-tuple of cocharacters $(\mu_1,\ldots,\mu_e)$ of $\widetilde{G}$ then, after recalling that

$$\widetilde{G} \cong \prod_{j=1}^f G\otimes_{k,\varphi^i}\mathbb{F}$$

where f denotes the degree of $k/\mathbb{F}_p$, we can also view a Hodge type as a tuple μ_{ij} of cocharacters of G for $1 \leq i \leq e$ and $1 \leq j \leq f$.

Proposition 18.1. *Let $\mathfrak{M} \in Z_{G,\mu,\mathbb{F}}(A)$ with A any finite type $\mathbb{F}$-algebra and μ a Hodge type satisfying*

$$\sum_{i=1}^e\langle\alpha^\vee,\mu_{ij}\rangle \leq p+e-1$$

for each root $\alpha^\vee$ of G and for each $1 \leq j \leq f$. Assume there is a $1 \leq j \leq f$ with the inequality strict for every $\alpha^\vee$. Then $\mathfrak{M}$ admits a unique crystalline G_K-action.

Before giving the proof we explain the propositions significance for us:

Corollary 18.2. *Assume μ is as in Proposition 18.1. Then the factorisation $Y_G^\mu\otimes_{\mathcal{O}}\mathbb{F} \to Z_{G,\mu,\mathbb{F}}$ from Theorem 17.1 is a closed immersion.*

Proof. Proposition 18.1 implies that the morphism $Y_G \otimes_{\mathcal{O}} \times_{Z_G} Z_{\mu,G,\mathbb{F}} \to Z_{\mu,G,\mathbb{F}}$ is an isomorphism. Since Y_G^μ is a closed subfunctor of Y_G it follows that

$$Y_G^\mu \times_{Z_G} Z_{G,\mu,\mathbb{F}} \to Z_{G,\mu,\mathbb{F}}$$

is a closed immersion. But Theorem 17.1 implies $Y_G^\mu \times_{Z_G} Z_{G,\mu,\mathbb{F}} = Y_G^\mu \otimes_{\mathcal{O}} \mathbb{F}$ so the corollary follows. □

Proof of Proposition 18.1. The claimed uniqueness of the G_K-action means that it will commute with any descent datum on $\mathfrak{M}$. Therefore, it suffices to prove the proposition after pulling $\mathfrak{M}$ back along an fppf cover of A. This allows us to assume that $\mathfrak{M}$ admits a trivialisation ι over $\operatorname{Spec} \mathfrak{S}_A$. The proposition then follows from the assertion that there exists a unique continuous cocycle

$$c : G_K \to U_{\mathrm{inf},A} := \ker\left(G(A_{\mathrm{inf},A}) \to G(A_{\mathrm{inf},A}/u\varphi^{-1}(\mu))\right)$$

satisfying $c(\sigma)\sigma(C) = C\varphi(c(\sigma))$ for each $\sigma \in G_K$ and $C = C_{\mathfrak{M},\iota}$. We will show how this is the case when C satisfies the two properties:

(1) For each $\sigma \in G_K$ one has $C\sigma(C^{-1}) \in U_{\mathrm{inf},A}$.
(2) The σ-conjugation operator $x \mapsto Cx\sigma(C^{-1})$ is such that $\mathrm{Ad}_\sigma(C) \circ \varphi$ stabilises $\widetilde{\mathfrak{g}} \otimes_{\mathcal{O}} u\varphi^{-1}(\mu)A_{\mathrm{inf},A}$ and is a topologically nilpotent. Here $\widetilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G})$ and φ on $\widetilde{g} \otimes_{\mathcal{O}} A_{\mathrm{inf},A}$ is the semi-linear extension of the trivial Frobenius on G.

These two properties will imply that, for $\sigma \in G_K$,

$$(\mathrm{Ad}_\sigma(C) \circ \varphi)^n (C\sigma(C^{-1}) \in U_{\mathrm{inf},A}$$

and the sequence converges to an element $c(\sigma)$, defining a continuous cocycle as desired. These claims can be checked after composing with a faithful representation $G \to \mathrm{GL}_n$ and, writing C also for its image in $\mathrm{GL}_n(A_{\mathrm{inf},A}[E(u)^{-1}])$, we are left showing that the difference

$$\begin{aligned}(\mathrm{Ad}_\sigma(C) \circ \varphi)^n(\sigma(C)C^{-1}) &- (\mathrm{Ad}_\sigma(C) \circ \varphi)^{n-1}(\sigma(C)C^{-1}) \\ &= (\mathrm{Ad}_\sigma(C) \circ \varphi)^{n-1}(\sigma(C)C^{-1})\left(C\varphi(C)\varphi(\sigma(C)^{-1})\sigma(C)^{-1} - 1\right)\end{aligned}$$

is contained in $u\varphi^{-1}(\mu)\,\mathrm{Mat}(A_{\mathrm{inf},A})$ for each n and converges to zero as $n \to \infty$. Now (1) implies $C\varphi(C)\varphi(\sigma(C)^{-1})\sigma(C)^{-1} - 1$ is a matrix with entries in $u\varphi^{-1}(\mu)A_{\mathrm{inf},A}$ and (2) implies that $\mathrm{Ad}_\sigma(C) \circ \varphi$ sends this matrix to another with entries in $u\varphi^{-1}(\mu)A_{\mathrm{inf},A}$, and that the action of this operator on the matrix is topologically nilpotent. Thus, the claimed convergence holds. For uniqueness, suppose $d(\sigma)$ is another such cocycle. Continuing to write $d(\sigma)$ and $c(\sigma)$ for their images under $G \to \mathrm{GL}_n$ we see that $d(\sigma) - c(\sigma)$ is fixed by $\mathrm{Ad}_\sigma(C) \circ \varphi$ and also that the action of $\mathrm{Ad}_\sigma(C) \circ \varphi$ is topologically nilpotent on this element. Therefore $d(\sigma) = c(\sigma)$. It remains to check that conditions (1) and (2) hold whenever $\mathfrak{M} \in Z_{G,\mu,\mathbb{F}}(A)$.

Verifying condition (2). Condition (2) will be a consequence of the fact that $\Psi(\mathfrak{M},\iota) \in Y_{\widetilde{G},\leq\mu}$ and the bound on μ. Indeed, $\Psi(\mathfrak{M},\iota) \in Y_{\widetilde{G},\leq\mu}$ ensures that

$$u^{h_j}\,\mathrm{Ad}(C) : \mathfrak{g}_j \mapsto \mathfrak{g}_j \otimes_{\mathcal{O}} \mathfrak{S}_A$$

where $h_j = \max \sum_{i=1}^e \langle \alpha^\vee, \mu_{ij}\rangle$ for $j = 1, \dots, f$ and $\mathfrak{g}_j = \mathrm{Lie}(G)$ denotes the Lie algebra of the j-th factor in $\widetilde{G} \cong \prod_{i=1}^f G \times_{W(k),\varphi} W(k)$. Thus $u^{h_j}\,\mathrm{Ad}_\sigma(C)$ sends $\mathfrak{g}_j$ into $\mathfrak{g}_j \otimes_{\mathcal{O}} A_{\mathrm{inf},A}$ also. Since φ on $\widetilde{g} \otimes_{\mathcal{O}} A_{\mathrm{inf},A}$ restricts to the map $\mathfrak{g}_j \otimes_{\mathcal{O}} A_{\mathrm{inf}} \to \mathfrak{g}_{j+1} \otimes_{\mathcal{O}} A_{\mathrm{inf}}$

which is the identity on the first factor and the Frobenius on the second, the operator $(\mathrm{Ad}(C)\circ\varphi)^n$ acts as

$$\begin{aligned}\mathfrak{g}_j \otimes_{\mathcal{O}} u\varphi^{-1}(\mu)A_{\mathrm{inf},A} &\xrightarrow{\mathrm{Ad}(C)\circ\varphi} \mathfrak{g}_{j+1}\otimes_{\mathcal{O}} u^{p-h_{j+1}}\mu A_{\mathrm{inf},A}\\ &\xrightarrow{\mathrm{Ad}(C)\circ\varphi} \mathfrak{g}_{j+1}\otimes_{\mathcal{O}} u^{p^2-ph_{j+1}-h_{j+2}}\varphi(\mu)A_{\mathrm{inf},A}\\ &\dots\\ &\xrightarrow{\mathrm{Ad}(C)\circ\varphi} \mathfrak{g}_{j+n}\otimes_{\mathcal{O}} u^{p^n-p^{n-1}h_{j+1}-p^{n-2}h_{j+2}-\ldots-h_{j+n}}\varphi^n(\mu)A_{\mathrm{inf},A}\end{aligned}$$

Since A is an $\mathbb{F}$-algebra we have $\varphi^n(\mu)A_{\mathrm{inf},A} = u^{ep^n/(p-1)}A_{\mathrm{inf},A}$ (see [Fon94b, 5.1.2]) so (2) is equivalent to asking that

$$p^n\left(\frac{e}{p-1}+1\right) - p^{n-1}h_{j+1} - p^{n-2}h_{j+2} - \ldots - h_{j+n} \geq \frac{p-1+e}{p-1}$$

for $n \geq 0$ and that this sequence converges to ∞ as $n \to \infty$. That this is the case under the assumptions on the h_j is an easy computation.

Verifying condition (1). Condition (1) will be a consequence of the fact that $\Psi(\mathfrak{M},\iota) \in M_\mu$, and holds without any assumption on μ. In fact, M_μ is contained inside a closed subscheme $\mathrm{Gr}_{\widetilde{G}}^{\nabla_\sigma} \subset \mathrm{Gr}_{\widetilde{G}}$ whose A-valued points, for any p-adically complete $\mathcal{O}$-algebra A of topologically finite type, consists of those $(\mathcal{E},\iota) \in \mathrm{Gr}_{\widetilde{G}}^{\nabla_\sigma}(A)$ for which there exists an fpcq cover $A' \to A$ trivialising $\mathcal{E}$ so that

$$(\mathcal{E},\iota)\otimes_A A' = (\mathcal{E}^0, C) \Leftrightarrow C\sigma(C)^{-1} \in U_{\mathrm{inf},A'}$$

for every $\sigma \in G_K$. That this condition is closed, and that $M_\mu \subset \mathrm{Gr}_{\widetilde{G}}^{\nabla_\sigma}$ easily reduce, after choosing a faithful representation, to the case of GL_n, where they are proved in [Bar21, 7.4] and [Bar21, 7.6]. □

19. Cycle inequalities

Now we can prove the main theorem:

Theorem 19.1. *Assume that G admits a twisting element ρ and let $\overline{\rho} : G_K \to G(\mathbb{F})$ be a continuous homomorphism. Let μ be a Hodge type with each μ_i strictly dominant. Suppose also that*

$$\sum_{i=1}^{e}\langle \alpha^\vee, \mu_i\rangle \leq p$$

for each root $\alpha^\vee$ of $\widetilde{G}$. Then, as $\sum_{i=1}^e \dim \widetilde{G}/P_{\mu_i}$-dimensional cycles inside of $\operatorname{Spec} R^\square_{\overline{\rho}} \otimes_{\mathcal{O}} \mathbb{F}$, one has

$$[\operatorname{Spec} R^{\square,\mathrm{cr},\mu}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathbb{F}] \leq \sum_\lambda m_\lambda [\operatorname{Spec} R^{\square,\mathrm{cr},\widetilde{\lambda}}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathbb{F}]$$

where

- *The $\leq$ indicates that the difference is an effective cycle, i.e. a $\mathbb{Z}_{\geq 0}$-linear combination of integral closed subschemes.*
- *The sum runs over dominant cocharacters λ of $\widetilde{G}$.*
- *$\widetilde{\lambda}$ denotes the Hodge type given by the e-tuple $(\lambda+\rho,\rho,\ldots,\rho)$.*

- m_λ *denotes the multiplicity of* $W(\lambda)$ *inside* $\otimes_{i=1}^e W(\mu_i - \rho)$. *It follows from* [Jan03, 5.6] *and* [Her09, 3.10] *that, due to the bound on* μ, m_λ *can equivalently be defined as the multiplicity of the representation of* $G(\mathbb{F})$ *obtained from the* $\mathbb{F}$*-valued points of* $W(\lambda)$ *inside that induced from the* $\mathbb{F}$*-valued points in* $\otimes_{i=1}^e W(\mu_i - \rho)$.

In the proof we use the standard functoriality of groups of cycles, namely the existence of a pullback homomorphism along flat morphisms and the pushforward along proper morphisms. See for example [Sta17, 02R3, 02RA].

Proof. First, we can assume $e > 1$ since when $e = 1$ the theorem is vacuous. As a consequence the inequality $\sum_{i=1}^e \langle \alpha^\vee, \mu_i \rangle \leq p$ ensures that Corollary 18.2 is applicable.

Theorem 12.1 gives an identity of cycles $[M_\mu \otimes_{\mathcal{O}} \mathbb{F}] = \sum_\lambda m_\lambda [M_{\widetilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}]$. For sufficiently large N we can pull this identity back along the formally smooth morphism Ψ_N giving an equality

$$[\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} (M_\mu \otimes_{\mathcal{O}} \mathbb{F})/_{\varphi} U_{G,N}] = \sum_\lambda m_\lambda [\widetilde{Z}_G \times_{\mathrm{Gr}_{\widetilde{G}}} (M_{\widetilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F})/_{\varphi} U_{G,N}]$$

of $\dim \mathcal{G}_{G,N} + \sum_{i=1}^e \dim \widetilde{G}/P_{\mu_i}$-dimensional cycles. This identity then descends to an identity

$$[Z_{\mu,G,\mathbb{F}}] = \sum_\lambda m_\lambda [Z_{\widetilde{\lambda},G,\mathbb{F}}]$$

of $\sum_{i=1}^e \dim \widetilde{G}/P_{\mu_i}$-dimensional cycles. Note that here we are discussing cycles inside an algebraic stack, as opposed to a scheme. In this case a cycle is again a formal linear combination of integral closed substacks, with the notion of multiplicity as discussed in [Sta17, 0DR4]. We also observe that, since $\mathcal{G}_{G,N}$ is smooth and irreducible, the irreducibility and generic reducedness of $M_{\widetilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}$ from Theorem 9.1 is shared by $Z_{\widetilde{\lambda},G,\mathbb{F}}$.

Now Corollary 18.2 implies that $[Y_G^\mu \otimes_{\mathcal{O}} \mathbb{F}] \leq [Z_{G,\mu,\mathbb{F}}]$. The irreducibility and generic reducedness of $Z_{G,\widetilde{\lambda},\mathbb{F}}$, together with the fact that $\dim Y_G^\mu \otimes_{\mathcal{O}} \mathbb{F} = \dim Z_{G,\mu,\mathbb{F}}$ implies that this is an equality when $\mu = \widetilde{\lambda}$. Therefore, we have

$$[Y_G^\mu \otimes_{\mathcal{O}} \mathbb{F}] \leq \sum_\lambda m_\lambda [Y_G^{\widetilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}]$$

Pulling this identity back along the formally smooth morphism from Lemma 16.6 gives

$$[\mathcal{L}_{\overline{\rho}}^{\mathrm{cr},\mu} \otimes_{\mathcal{O}} \mathbb{F}] \leq \sum_\lambda m_\lambda [\mathcal{L}_{\overline{\rho}}^{\mathrm{cr},\widetilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}]$$

Finally, pushing this identity along the proper morphism $\mathcal{L}_{\overline{\rho}}^{\mathrm{cr},\mu} \otimes_{\mathcal{O}} \mathbb{F} \to \operatorname{Spec} R_{\overline{\rho}}^{\square} \otimes_{\mathcal{O}} \mathbb{F}$ and using [Bar21, 3.3] to equate the pushforward of $[\mathcal{L}_{\overline{\rho}}^{\mathrm{cr},\mu} \otimes_{\mathcal{O}} \mathbb{F}]$ with $[\operatorname{Spec} R_{\overline{\rho}}^{\square,\mathrm{cr},\mu} \otimes_{\mathcal{O}} \mathbb{F}]$ proves the theorem. □

20. Monodromy and Galois

Here we give a proof of the following equivalent formulation of Proposition 17.8. For simplicity we work with $\mathbb{Z}_p$ coefficients but the extension to any coefficient ring which is finite and flat over $\mathbb{Z}_p$ is immediate.

Proposition 20.1. *Let $\mathfrak{M}$ denote the Breuil–Kisin module associated to a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\mathbb{Z}_p)$ and let N_∇ be the operator over $\partial = u\frac{d}{du}$ on $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ induced from the φ-equivariant identification*

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\lambda}] \cong D(\rho) \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\lambda}]$$

described in 17.5. If $N_\nabla(\iota) = \iota N$ for an $\mathfrak{S}$-basis ι of $\mathfrak{M}$ then $N \in \frac{u}{p}\,\mathrm{Mat}(S_{\max})$ for $S_{\max} \coloneqq W(k)[[u, \frac{u^e}{p}]]$.

As explained in Remark 17.9, the results of this section are not strictly speaking necessary for this paper but we still think they may be useful to help orient the reader.

20.2. The ideas go back to [GLS14], with the new insight being that improved bounds can be achieved by replacing Fontaine's crystalline period ring B_{crys} with a better behaved period ring $B_{\max}$. This ring is defined in [Col98, §III] by considering the subring $A_{\max}$ of B_{dR}^+ consisting of elements which can be expressed as

$$\sum_{n\geq 0} x_n \left(\frac{\nu}{p}\right)^n$$

for ν any element generating the kernel of usual surjection $\Theta : A_{\inf} \to \mathcal{O}_C$ and $x_n \in A_{\inf}$ a sequence converging p-adically to zero. Note that $E(u)$ is one such generator of this kernel. Then $B_{\max}^+ = A_{\max}[\frac{1}{p}]$ and $B_{\max} = B_{\max}^+[\frac{1}{t}]$ for $t \coloneqq \log([\epsilon]) = \sum_{n\geq 0}(-1)^n\frac{([\epsilon]-1)^n}{n}$. The essential property that we will need is:

Lemma 20.3. *Recall that $\mathcal{O}^{\mathrm{rig}}$ denotes the ring of power series in $K_0[[u]]$ converging on the open unit disk, and $\lambda \coloneqq \prod_{n\geq 0}\varphi^n(\frac{E(u)}{E(0)})^n$. Then the inclusion of $\mathfrak{S} \to A_{\inf}$ extends to an embedding of $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}] \to B_{\max}$ so that*

$$\mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\lambda}] \cap A_{\max} \subset S_{\max}$$

Proof. An easy computation shows that each $\frac{\varphi^n(E(u))}{p}$ is invertible in $S_{\max}$, and so it suffices to show $\mathcal{O}^{\mathrm{rig}} \cap A_{\max} \subset S_{\max}$. Any $f \in \mathcal{O}^{\mathrm{rig}}$ can be expressed uniquely as

$$f = \sum(\tfrac{E(u)}{p})^n q_n$$

with $q_n \in K_0[u]$ polynomials of degree $< e$ converging p-adically to zero and we have $f \in S_{\max}$. We claim that $f \in A_{\max}$ if and only if each $q_n \in W(k)[u]$. This will prove the proposition because it will imply $f \in S_{\max}$. To see this we use a result of Colmez. Recall that Θ extends to a surjection $\Theta : B_{\mathrm{dR}}^+ \to C$ and, following [Col98, §V.3], we call an element $x \in B_{\mathrm{dR}}^+$ flat if $\theta(x) \neq 0$ and if $x \in p^{w(x)}A_{\inf}$ where $w(x)$ denotes the integer part of $v_p(\Theta(x))$. We also say zero if flat. If $q_n = \sum_{i=0}^{e-1} a_i u^i$ is non-zero then $\Theta(q_n) = \sum_{i=0}^{e-1} a_i \pi^i$ is non-zero and $w(\Theta(q_n)) = \min v_p(a_i)$. Thus $q_n \in p^{w(\Theta(q_n))}A_{\inf}$ and so each q_n is flat. Colmez shows in [Col98, Lemme V.3.1] that if $x \in B_{\mathrm{dR}}^+$ can be expressed as a sum $\sum_{n\geq 0} y_n(\frac{\nu}{p})^n$ with $\nu \in A_{\inf} \cap \ker\Theta$ a generator and $y_n \in B_{\mathrm{dR}}^+$ flat, then $x \in A_{\max}$ if and only if $w(y_n) \geq 0$ and converges to ∞. Since $E(u)$ is one possible generator of $\ker\Theta$ this gives the result. □

Combining Lemma 20.3 with the following gives Proposition 20.1.

Proposition 20.4. *With notation as in Proposition 20.1 one has $N \in \frac{u}{p\lambda}\,\mathrm{Mat}(A_{\max})$.*

20.5. To prove this first observe that $A_{\max}$ has a natural Frobenius φ and a φ-equivariant G_K-action extending that on $A_{\inf}$. Furthermore, one has $\varphi(B_{\max}) \subset B_{\mathrm{crys}} \subset B_{\max}$ which shows that $B_{\max}$ can be used as a replacement for the crystalline period ring B_{crys}. This means that there are φ-equivariant identifications

$$\rho \otimes_{\mathbb{Z}_p} B_{\max} \cong D(\rho) \otimes_{K_0} B_{\max} \cong \mathfrak{M} \otimes_{\mathfrak{S}} B_{\max} \tag{20.6}$$

with the first being G_K-equivariant for the trivial action of G_K on $D(\rho)$. Here the second isomorphism is the base-change of the φ-equivariant isomorphism

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\lambda}] \cong D(\rho) \otimes_{K_0} \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\lambda}] \tag{20.7}$$

described in 17.5, while the composite is obtained from the identification

$$\mathfrak{M} \otimes_{\mathfrak{S}} W(C^\flat) \cong \rho \otimes_{\mathbb{Z}_p} W(C^\flat) \tag{20.8}$$

in Theorem 16.2, after applying [BMS18, 4.26] to descend this isomorphism to $A_{\inf}[\frac{1}{\mu}]$, and then base-changing to $B_{\max}$.

The key to proving Proposition 20.4 is to relate, inside of (20.6), the G_K-action coming ρ with the operator N_∇ coming from $D(\rho)$:

Lemma 20.9. *For any $\sigma \in G_K$ and $m \in \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ one has*

$$(\sigma - 1)(m) = \sum_{n \geq 1} N_\nabla^n(m) \otimes \frac{(-\log([\epsilon(\sigma)]))^n}{n!}$$

Conversely, if $\sigma \in G_K$ acts trivially on $\mathbb{Z}_p(1) = \varprojlim \mu_{p^n}(C)$ (i.e. if the cyclotomic character χ_{cyc} is trivial on σ) then

$$N_\nabla(m) = \frac{-1}{\log([\epsilon(\sigma)])} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\sigma - 1)^n}{n}(m)$$

where $\epsilon(\sigma) \in \mathbb{Z}_p(1) = \sigma(u)/u$.

Here convergence of the sums is taken with respect to the topology on $B^+_{\max}$ with basis of open neighbourhoods of 0 given by $p^n A_{\max}$. Since $A_{\max}$ is p-adically complete so is $B^+_{\max}$ for this topology.

Proof. It suffices to check these identities for $m = d \otimes f$ for $d \in D(\rho)$ and $f \in \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$. Since $(\sigma - 1)(d) = N_\nabla(d) = 0$ the lemma reduces to the claim that when $\chi_{\mathrm{cyc}}(\sigma) = 1$

$$\frac{-1}{\log([\epsilon(\sigma)])} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\sigma - 1)^n}{n}(f)$$

converges in $B^+_{\max}$ to $\partial(f)$ and, for any $\sigma \in G_K$,

$$\sum_{n \geq 1} \partial^n(f) \otimes \frac{(-\log([\epsilon(\sigma)]))^n}{n!}$$

converges in $B^+_{\max}$ to $(\sigma - 1)(f)$. It suffices to check either claim when $f = u^i$. For the first note that if $\chi_{\mathrm{cyc}}(\sigma) = 1$ then $(\sigma - 1)^n(u^i) = u^i([\epsilon(\sigma)] - 1)^n$ for all n. Therefore the claimed convergence follows from the easy observation $[\epsilon(\sigma)] - 1 \in pA_{\max}$ when $p > 2$ and that, when $p = 2$, instead has $[\epsilon(\sigma)] - 1 = ([\epsilon(\sigma)]^{1/2} - 1)([\epsilon(\sigma)]^{1/2} + 1) \in 4A_{\max}$. For the second claim we note that, since $\partial^n(f) = (-i)^n u^i$ and so $\sum_{n \geq 1} \partial^n(f) \otimes \frac{(-\log([\epsilon(\sigma)]))^n}{n!} = \exp(\log([\epsilon(\sigma)]))$. By the same argument as above, this converges to $u^i[\epsilon(\sigma)] = \sigma(f)$. $\square$

Next we prove the divisibility of the G_K-action asserted in Theorem 16.2. Actually, we need something a little stronger:

Proposition 20.10. *If $m \in \mathfrak{M}$ and $\sigma \in G_K$ then*
$$(\sigma - 1)^n(m) \in \mathfrak{M} \otimes_{\mathfrak{S}} u\varphi^{-1}(\mu)^n A_{\mathrm{inf}}$$
for $n = 1$. If additionally $\chi_{\mathrm{cyc}}(\sigma) = 1$ then this is true for all $n \geq 1$.

Here we will use that the topology on $B^+_{\max}$ (in contrast to that on B^+_{crys}) is well behaved. More precisely, one has [Col98, Proposition III.2.1] which implies that any principal ideal in $B^+_{\max}$ is closed.

Proof. We show the equivalent assertion that $(\sigma - 1)^n(\varphi(m)) \in \mathfrak{M}^\varphi \otimes_{\mathfrak{S}} u^p\mu^n A_{\mathrm{inf}}$ for $\mathfrak{M}^\varphi$ the image of $\varphi^*\mathfrak{M}$ in $\mathfrak{M}[\frac{1}{E(u)}]$ under the Frobenius. Iterating the formula in Lemma 20.9 shows that $(\sigma - 1)^n(\varphi(m))$ can be expressed as
$$\sum_{j=n}^{\infty} \left(\sum_{j_1+\ldots+j_n=j, j_i\geq 1} N^j_\nabla(m) \otimes \frac{(-\log([\epsilon(\sigma)]))^j}{j_1!\ldots j_n!} \right) \tag{20.11}$$
for $n = 1$ and any $\sigma \in G_K$ and, if $\chi_{\mathrm{cyc}}(\sigma) = 1$, for all $n \geq 1$. As explained in 17.5, (20.7) arises from an identification $\mathfrak{M}^\varphi \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}] \cong D(\rho) \otimes_{K_0} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$. This means that $N^j_\nabla(\varphi(m)) \in \mathfrak{M}^\varphi \otimes_{\mathfrak{S}} u\mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$ for each $j \geq 1$ and so each term of (20.11) is contained in $u^pt^jB^+_{\max}$. Since all principal ideals in $B^+_{\max}$ are closed it follows that the entire sum is contained in $\mathfrak{M}^\varphi \otimes_{\mathfrak{S}} u^pt^nB^+_{\max}$ also.

On the other hand, since (20.8) descends to an isomorphism over $A_{\mathrm{inf}}[\frac{1}{\mu}]$, we also know that $(\sigma - 1)(\varphi(m)) \in \mathfrak{M}^\varphi \otimes_{\mathfrak{S}} A_{\mathrm{inf}}[\frac{1}{\mu}]$. The proposition will therefore follow from the assertion that
$$A_{\mathrm{inf}}[\tfrac{1}{\mu}] \cap u^pt^nB^+_{\max} = u^p\mu^nA_{\mathrm{inf}}$$
To prove this first note that $\frac{t}{\mu}$ is a unit in $A_{\max}$ by [Col98, Lemme III.3.9]. Therefore, we need to show that if $a \in A_{\mathrm{inf}} \cap \mu^nB^+_{\max}$ then $a \in \mu^nA_{\mathrm{inf}}$ and if $a \in A_{\mathrm{inf}} \cap u^nB_{\max}$ then $a \in u^nA_{\mathrm{inf}}$. The first claim follows from the fact [Fon94a, Proposition 5.1.3] that μ generates the ideal in consisting of those $x \in A_{\mathrm{inf}}$ with $\varphi^n(x) \in \ker\theta$ for all $n \geq 0$. For the second claim we use [Liu13, Lemma 3.2.2] which shows $u^nB^+_{\mathrm{crys}} \cap A_{\mathrm{inf}} = u^nA_{\mathrm{inf}}$. Since $\varphi(B^+_{\max}) \subset B_{\mathrm{crys}} \subset B_{\max}$, if $b \in u^nB^+_{\max} \cap A_{\mathrm{inf}}$ then $\varphi(b) \in u^{pn}B^+_{\mathrm{crys}} \cap A_{\mathrm{inf}} = u^{pn}A_{\mathrm{inf}}$. Thus $b \in u^nA_{\mathrm{inf}}$, as required. □

Finally we can prove:

Proof of Proposition 20.4. The assumption that $K_\infty \cap K(\mu_{p^\infty}) = K$ ensures that $\sigma \in G_K$ can be found with $\epsilon(\sigma)$ equal the fixed generator $\epsilon \in \mathbb{Z}_p(1)$ and $\chi_{\mathrm{cyc}}(\sigma) = 1$. For such a σ we have
$$N_\nabla(m) = \frac{-1}{t} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\sigma - 1)^n}{n}(m)$$
for any $m \in \mathfrak{M}$. By Proposition 20.10 we know $(\sigma - 1)^n(m) \in \mathfrak{M} \otimes_{\mathfrak{S}} u\varphi^{-1}(\mu)A_{\mathrm{inf}}$ for each $n \geq 1$. We are going to show that each term in the above sum, and hence the sum itself, is contained $\mathfrak{M} \otimes_{\mathfrak{S}} \frac{u\varphi^{-1}(\mu)}{t}A_{\max}$.

For this claim it suffices to show that $\varphi^{-1}(\mu)^{n-1} \in nA_{\max}$. Since $\varphi^{-1}(\mu)^p \equiv \mu$ modulo pA_{inf} it follows that $\alpha := \frac{\varphi^{-1}(\mu)^p}{p} - \frac{\mu}{p} \in A_{\mathrm{inf}}$. Since $\varphi^n(\alpha) \in \ker\theta$ for all $n \geq 1$

we know that $\varphi(\alpha)$ is divisible by μ in A_{inf}. Hence $\frac{\alpha}{\varphi^{-1}(\mu)} = \frac{\varphi^{-1}(\mu)^{p-1}}{p} - \frac{\mu}{\varphi^{-1}(\mu)p} \in A_{\mathrm{inf}}$. Since $\frac{\mu}{\varphi^{-1}(\mu)}$ generates the kernel of Θ it follows that $\frac{\varphi^{-1}(\mu)^{p-1}}{p} \in A_{\max}$, and so the claim holds when $n = p$. For general n, we write $n = p^s m$ for m coprime to p. Since $p^s - 1 = (p-1)(1 + p + \ldots + p^{s-1})$ we have $n - 1 \geq p^s - 1 \geq (p-1)s$ and so $\frac{\varphi^{-1}(\mu)^{n-1}}{p^s} \in A_{\max}$ which proves the claim.

It remains only to show that $\frac{\varphi^{-1}(\mu)}{t} A_{\max} = \frac{1}{p\lambda} A_{\max}$. We showed above that μ generates the same ideal of $A_{\max}$ as t so this is equivalent to showing that $\frac{\mu}{\varphi^{-1}(\mu)p}$ and λ generate the same ideal. As $\frac{\mu}{\varphi^{-1}(\mu)}$ and $E(u)$ generate the same ideal in A_{inf} this is equivalent to asking that $\varphi(\lambda)$ is a unit in $A_{\max}$. But this is clear because $\frac{\varphi^n(E(u))}{E(0)} - 1$ is topologically nilpotent in $A_{\max}$ for $n \geq 1$. $\square$

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