GKLO REPRESENTATIONS OF TWISTED YANGIANS IN TYPE AI AND QUANTIZATIONS OF SYMMETRIC QUOTIENTS OF THE AFFINE GRASSMANNIAN

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Abstract. We construct an analogue of Gerasimov–Kharchev–Lebedev–Oblezin (GKLO) representations for twisted Yangians of type AI, using the recently found current presentation of these algebras due to Lu, Wang and Zhang. These new representations allow us to define interesting truncations of twisted Yangians, which, in the spirit of Ciccoli–Drinfeld–Gavarini quantum duality, reflect the Poisson geometry of homogeneous spaces. As our main result, we prove that a truncated twisted Yangian quantizes a scheme supported on quotients of transverse slices in the affine grassmannian.

Contents

1. Introduction

In [GKLO05], Gerasimov, Kharchev, Lebedev and Oblezin (GKLO) constructed a family of infinite-dimensional representations of the Yangian using explicit difference

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operators. These GKLO representations were later generalized to the shifted Yangian setting and used to produce quantisations of transverse slices to Schubert varieties in the affine Grassmannian [KWWY14, BFN19]. In this paper we develop an analogue of that picture for twisted Yangians of type AI. Concretely, we construct explicit GKLO-style representations of these twisted Yangians and realise the resulting truncations as quantisations of loci inside quotients of the thick affine Grassmannian.

Twisted Yangians, introduced by Olshanski [Ols92], arise in mathematical physics from the Yang–Baxter and reflection equations as algebras controlling the symmetries of integrable systems with boundaries. From an algebraic point of view, they appear naturally in the context of Gelfand–Tsetlin theory for simple Lie algebras of classical types [Mol06]. Recently, they have also been realized as degenerations of affine quantum symmetric pair coideal subalgebras, or ıquantum groups [LWZ25]. Shifted versions of twisted Yangians, which we denote by ${}^{\mathsf{tw}}\mathbf{Y}_\mu$ for dominant $\mu \in X_*(T)$, were introduced in [TT24] building on the Drinfeld presentation from [LWZ23].

Setup. Throughout we take $G = \mathrm{SL}_n$ with diagonal torus T and write $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$. Let $N(T) \subset G$ denote the normaliser of T so that $W = N(T)/T$. We consider the thick affine grassmannian $\mathrm{Gr}^{\mathrm{thick}} = G((z^{-1}))/G[z]$ with the Poisson structure induced from the standard Manin triple for the loop algebra. If $\mathcal{G}_0 \subset G[[z^{-1}]]$ denotes the first congruence subgroup then, for any dominant (always considered relative to the standard upper triangular Borel) $\mu \in X_*(T)$, the $\mathcal{G}_0$-orbits Gr_μ through z^μ are known to be Poisson subschemes. Finally, we consider the involution

$$\tau : G((z^{-1})) \to G((z^{-1})), \qquad \tau(g) = g(-z)^t$$

where x^t denotes the transpose of x.

Poisson geometry. The first part of this paper analyses the Poisson geometry of the quotient spaces $K_0 \backslash \mathrm{Gr}_\mu$. This is done in Section 3 and the following summarises our main results.

Theorem 1.1. *Let $K \subset G((z^{-1}))$ denote the subgroup defined by $g(-z)^t = g(z)^{-1}$ and set $K_0 = K \cap \mathcal{G}_0$. Then*

(1) *K_0 is a coisotropic subgroup of $\mathcal{G}_0$ and so, for each $\mu \in X_*(T)$, the quotient $K_0 \backslash \mathrm{Gr}_\mu$ (which is representable by an affine scheme) inherits a Poisson structure from Gr_μ.*
(2) *The symplectic leaves inside $K_0 \backslash \mathrm{Gr}_\mu$ are connected components of the subschemes*

$$K_0 \backslash (Kxz^\eta \cap \mathrm{Gr}_\mu) \tag{1.1}$$

where $\eta \in X_*(T)$ and $x \in G$ is such that $x^t x \in N(T)$ represents an involution $w \in W$ with $\eta + w(\eta) \geq 2\mu$ and dominant. The subscheme (1.1) is uniquely determined by $\lambda = \eta + w(\eta)$, except when w has no fixed points, in which case *there are two such subschemes corresponding to two possible choices of x. If $\mu = 0$ then (1.1) is connected and hence a single symplectic leaf.*

(3) *Write $\mathcal{S}_{2\mu}^{\lambda}$ for the subscheme (1.1) when w has at least one fixed point and the union of the two distinct such subschemes otherwise. If $\mu = 0$ then the ideal of the reduced closed subscheme*

$$\mathcal{S}_{2\mu}^{\leq\lambda} = \bigcup_{\gamma\leq\lambda} \mathcal{S}_{2\mu}^{\lambda}$$

inside $K_0 \backslash \mathrm{Gr}_\mu$ can be described as the radical of an ideal Poisson generated by a set of explicit rational functions defined via trailing principal minors.

A central idea underlying the proof of these results is the existence of an isomorphism

$$K_0 \backslash \mathrm{Gr}_\mu \cong \mathrm{Gr}_{2\mu}^{\tau=1}$$

induced via $K_0 x \mapsto \tau(x)x$ for $\tau(g(z)) = g(-z)^t$. This identifies the quotient Poisson structure on the left hand side with that on the right hand side obtained via Dirac reduction (up to a factor of 2). It also identifies the subschemes $\mathcal{S}_{2\mu}^{\lambda}$ from Theorem 1.1 with the τ-fixed points inside the loci $\mathrm{Gr}_{2\mu}^{\lambda} = \mathrm{Gr}_{2\mu} \cap \mathrm{Gr}^{\lambda}$ with $\mathrm{Gr}^{\lambda} \subset \mathrm{Gr}^{\mathrm{thick}}$ the Schubert cell given as the $G[z]$-orbit through z^{λ}.

We expect that the restrictions to $\mu = 0$ in Theorem 1.1 are unnecessary. However, we are currently unable to prove this. The most significant obstruction arises in the proof of part (3) of Theorem 1.1.

GKLO-representations and quantisation. Our second main result shows the Poisson structures from Theorem 1.1 are quantised by twisted Yangians and their truncations. Write ${}^{\mathsf{tw}}\mathbf{Y}_\mu$ for the $\mathbb{C}[\hbar]$-form of the twisted Yangian shifted by a dominant $\mu \in X_*(T)$.

Theorem 1.2. *For each dominant $\mu \in X_*(T)$ there is an isomorphism of Poisson $\mathbb{C}$-algebras*

$$(1.2) \qquad {}^{\mathsf{tw}}\mathbf{Y}_\mu / \hbar\, {}^{\mathsf{tw}}\mathbf{Y}_\mu \cong \mathcal{O}(K_0 \backslash \mathrm{Gr}_\mu).$$

Furthermore,

(1) *For each dominant $\lambda \in X_*(T)$ we define quotients ${}^{\mathsf{tw}}\mathbf{Y}_\mu^\lambda$ of ${}^{\mathsf{tw}}\mathbf{Y}_\mu$ via twisted GKLO-representations. See Theorem 7.2.*

(2) *If $\mu = 0$ then the identification (1.2) induces a Poisson surjection*

$$ {}^{\mathsf{tw}}\mathbf{Y}_\mu^\lambda / \hbar\, {}^{\mathsf{tw}}\mathbf{Y}_\mu^\lambda \to \mathcal{O}(\mathcal{S}_{2\mu}^{\leq -w_0(2\lambda)})$$

which is an isomorphism up to nilpotent elements.

The twisted GKLO representations from part (1) of Theorem 1.2 are given in terms of the current, or 'new Drinfeld', presentation of the twisted Yangian, found recently by Lu, Wang and Zhang [LWZ23]. The generators commonly denoted as $b_i(u)$ act as sums of localized difference operators, via formulae which somewhat resemble the Gelfand–Tsetlin formulae for quantum symmetric pairs [GK91, LP25]. The shape of our formulae is motivated by the realization of the twisted Yangian in terms of Sklyanin minors. Namely, the corresponding Cartan generators, given by the principal Sklyanin minors, are required to act as *even* polynomials. This restriction determines the correct coefficients on the difference operators corresponding to the $b_i(u)$ generators. The presence of this extra symmetry is a new feature, absent from the original GKLO representations.

We also expect that the map in part (2) of Theorem 1.2 is actually an isomorphism, so that ${}^{\mathsf{tw}}\mathbf{Y}_\mu$ directly quantises $\mathcal{S}_{2\mu}^{\leq -w_0(2\lambda)}$, rather than a non-reduced scheme supported on this locus. This could be proved if one knew the Poisson ideal discussed in part (3) of Theorem 1.1 was reduced. In Conjecture 3.14 we formulate a conjecture in this direction and show how its validity implies an explicit description of the ideal defining the truncation ${}^{\mathsf{tw}}\mathbf{Y}_\mu^\lambda$ inside ${}^{\mathsf{tw}}\mathbf{Y}_\mu$.

Theorem 1.3. *Assume $\mu = 0$ and that Conjecture 3.14 holds. Then the surjection in part (2) of Theorem 1.2 is an isomorphism and ${}^{\mathsf{tw}}\mathbf{Y}_\mu^\lambda$ is the quotient of ${}^{\mathsf{tw}}\mathbf{Y}_\mu$ by a two sided ideal generated by elements $A_i^{(r)}$ for $r \geq r_i := \langle \omega_i, 2\lambda \rangle$ and $B_i^{(r_i+1)}$ for each $1 \leq i \leq n-1$.*

A concrete example of Ciccoli–Drinfeld–Gavarini duality. Theorem 1.2 can be viewed through the lens of the Drinfeld–Gavarini quantum duality principle [Dri87b, Gav02, Gav07]. For a Poisson–Lie group H with Lie algebra $\mathfrak{h}$, quantum duality asserts that the semiclassical limit of $U_\hbar(\mathfrak{h})$ is isomorphic, as a Poisson–Hopf algebra, to $\mathcal{O}(H^*)$, the algebra of functions on the dual Poisson–Lie group. Applied to $H = G[z]$ this identifies the semiclassical Yangian with functions on the dual Poisson group $\mathcal{G}_0$ and is the starting point for the results in [KWWY14]. Ciccoli–Gavarini [CG06, CG14] extended quantum duality to Poisson homogeneous spaces (quotients by coisotropic subgroups). If $N \subset H$ is coisotropic then the algebra of $N^\perp$-invariant functions on the dual Poisson group corresponds to the semiclassical limit of a (one-sided) coideal subalgebra. In our setting the subgroup $N = G[z]^\tau$ and its orthogonal complement give rise (via this principle) to twisted Yangian-type coideal algebras; Theorem 1.2 provides a concrete realisation of this correspondence in the split type AI case.

Remark 1.4. Just before the completion of our paper, another preprint on the same topic was independently released by Lu, Wang and Weekes [LWW25]. There is considerable intersection between the two papers. For example, we both introduce GKLO-style representations and study the Poisson geometry of fixed point loci in affine Grassmannian slices. Many aspects of [LWW25] are more general, e.g., they also construct GKLO representations for twisted Yangians of type AIII, consider generalized affine Grassmannian slices, and introduce the notion of an iCoulomb branch.

On the other hand, elements of our approach differ significantly from loc. cit. and appear to offer more control towards several outstanding open problems. For example, Theorem 1.2 provides the partial step towards Conjecture 8.13 of loc. cit. as described in Remark 8.14. In a similar spirit, our Conjecture 3.14 provides a direct approach towards Conjecture 3.10 of loc. cit. and presents the possibility of emulating the strategy devised in the untwisted setting in [KMWY18].

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2. Notation

2.1. **Basic setup.** Let $G = \mathrm{SL}_n$ viewed as an algebraic group over $\operatorname{Spec}\mathbb{C}$, with diagonal torus T and upper triangular Borel B. Also write $U^{\pm}$ respectively for the upper and lower triangular unipotent subgroups in G. Set $W = N(T)/T$ for $N(T) \subset G$ the normaliser of T and write $w_0 \in W$ for the longest element.

Set $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ and $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, which are dual via the evaluation pairing

$$\langle -, - \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}, \qquad (\beta^\vee, \lambda) \mapsto \langle \beta^\vee, \lambda \rangle = \beta^\vee(\lambda).$$

We use analogous notations for the diagonal torus $\widetilde{T} \subset \mathrm{GL}_n$, with canonical maps $X_*(T) \hookrightarrow X_*(\widetilde{T})$ and $X^*(\widetilde{T}) \twoheadrightarrow X^*(T)$.

We will use the index sets $\mathbb{I} = \{1, \cdots, n-1\}$ and $\tilde{\mathbb{I}} = \{1, \cdots, n\}$. Let $(a_{ij})_{i,j\in\mathbb{I}}$ be the Cartan matrix associated with G. Given $i \in \tilde{\mathbb{I}}$, we write $\epsilon_i^\vee \in X^*(\widetilde{T})$ for the character

$$\mathrm{diag}(t_1, \dots, t_n) \mapsto t_i$$

and ϵ_i for its dual under $\langle -, - \rangle$. Let Δ^+ be the set of all positive roots of G (relative to B), with simple roots $\alpha_i^\vee = \epsilon_i^\vee - \epsilon_{i+1}^\vee$ $(i \in \mathbb{I})$. We write $\omega_i \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ for the fundamental coweights, dual to $\alpha_i^\vee$, and $\omega_i^\vee \in X^*(T)$ for the fundamental weights. Let $X_*(T)^+ \subset X_*(T)$ denote the set of dominant coweights relative to B, so $\lambda \in X_*(T)^+$ iff $\langle \alpha_i^\vee, \lambda \rangle \geq 0$ for each $i \in \mathbb{I}$ and write $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of positive coroots. .

2.2. **Affine grassmannians.** Write $G((z^{-1}))$, $G[z]$, and $G[z, z^{-1}]$ for the ind-group schemes over $\operatorname{Spec}\mathbb{C}$ with A-valued points given respectively by $G(A((z^{-1})), G(A[z])$ and $G(A[z, z^{-1}])$. We use similar notation when G is replaced by another affine group scheme over $\operatorname{Spec}\mathbb{C}$. We then consider the fpqc quotients

$$\mathrm{Gr}^{\mathrm{thick}} = G((z^{-1}))/G[z], \qquad \mathrm{Gr}^{\mathrm{thin}} = G[z, z^{-1}]/G[z]$$

the first of which is representable by a scheme, and the latter by an ind-scheme. For any coweight $\lambda \in X_*(T)$ we write $z^\lambda \in \mathrm{Gr}^{\mathrm{thick}}$ for the image of $\lambda(z) \in G((z^{-1}))$ and if

λ is dominant we write Gr^λ for the $G[z]$-orbit through this point, with closure $\mathrm{Gr}^{\leq\lambda}$. We also consider the subgroup

$$\mathcal{G}_0 \subset G((z^{-1}))$$

whose A-points, for any $\mathbb{C}$-algebra A, consist of matrices in $1 + z^{-1}\,\mathrm{Mat}(A[[z^{-1}]])$. Recall that if $\mathcal{U}_0^\pm = \mathcal{G}_0 \cap U^\pm((z^{-1}))$ and $\mathcal{T}_0 = \mathcal{G}_0 \cap T((z^{-1}))$ then multiplication defines an isomorphism

$$\mathcal{U}_0^+ \times \mathcal{T}_0 \times \mathcal{U}_0^- \to \mathcal{G}_0 \tag{2.1}$$

(indeed multiplication $U^+ \times T \times U^- \to G$ is known to be an open immersion whose image is the open locus defined by the non-vanishing of the principal minors). For $\mu \in X_*(T)^+$ set $\mathrm{Gr}_\mu \subset \mathrm{Gr}^{\mathrm{thick}}$ equal the $\mathcal{G}_0$-orbit through z^μ. Finally, for any $1 \leq i \leq n$ and any pair of i-tuples $I, J \subset \{1, \dots, n\}$ we write

$$\Delta_{IJ} \in \mathcal{O}(\mathcal{G}_0)[[z^{-1}]]$$

for the series valued function whose value on $g \in \mathcal{G}_0$ is its IJ-th minor. Write $\Delta_{IJ}^{(r)} \in \mathcal{O}(\mathcal{G}_0)$ for the coefficient of z^{-r} in this series.

2.3. **Commutators.** We use the following notation for commutators: $[a, b] = ab - ba$ and $[a, b]_+ = ab + ba$.

3. Symmetric quotients of the affine Grassmannian

3.1. **Poisson structures on loop groups.** Following [KWWY14] we equip $G((z^{-1}))$ with the Poisson structure induced by the Manin triple $(\mathfrak{g}((z^{-1})), \mathfrak{g}[z], z^{-1}\mathfrak{g}[[z^{-1}]])$, with the pairing

$$(x, y) = \mathrm{Res}_{z=0}\,\mathrm{Trace}(xy)$$

i.e., the coefficient of z^{-1} in $\mathrm{Trace}(xy)$. In [KWWY14, Proposition 2.13] (specialised to the case $G = \mathrm{SL}_n$) the resulting Poisson bracket $\{-,-\}$ is computed as:

Lemma 3.1. *Recall the series valued functions $\Delta_{IJ} = \Delta_{IJ}(z)$ from Section 2.2. Then*

$$\{\Delta_{IJ}(u), \Delta_{KL}(v)\} = \frac{1}{u-v} \sum_{1 \leq p,q \leq n} \left(\epsilon_{pq}^{J,L} \Delta_{IJ^{(p\rightsquigarrow q)}}(u) \Delta_{KL^{(q\rightsquigarrow p)}}(v) - \epsilon_{qp}^{I,K} \Delta_{I^{(q\rightsquigarrow p)}J}(u) \Delta_{K^{(p\rightsquigarrow q)}L}(v)\right)$$

where

- *$J^{(p\rightsquigarrow q)}$ denotes the tuple with $p \in J$ replaced by q. Likewise for $L^{(q\rightsquigarrow p)}, I^{(q\rightsquigarrow p)}$, and $K^{(p\rightsquigarrow q)}$.*
- *$\epsilon_{pq}^{J,L} = 0$ if $p \notin J$ or $q \notin L$ and otherwise equals ± 1 according to the sign of the permutations reordering $J^{(p\rightsquigarrow q)}$ and $L^{(q\rightsquigarrow p)}$ into ascending order.*

Both $G[z]$ and $\mathcal{G}_0$ appear as Poisson subgroups of $G((z^{-1}))$ and the quotient $\mathrm{Gr}^{\mathrm{thick}} = G((z^{-1}))/G[z]$ inherits a Poisson structure from that on $G((z^{-1}))$.

Lemma 3.2. *Recall* $\mathcal{U}_0^- = \mathcal{G}_0 \cap U^-((z^{-1}))$*. For* $\mu \in X_*(T)$ *dominant, set* $\mathcal{U}^{-,\mu} = \mathcal{G}_0 \cap z^\mu G[z] z^{-\mu}$ *and* $\mathcal{U}_\mu^- = z^\mu \mathcal{U}_0^- z^{-\mu}$*. Then both are subgroups of* $\mathcal{U}_0^-$ *and:*

(1) $\mathcal{U}^{-,\mu}$ *is a coisotropic subgroup in* $\mathcal{G}_0$*, and so* $\mathcal{G}_0 \to \mathcal{G}_0/\mathcal{U}^{-,\mu}$ *is a Poisson quotient.*
(2) *The quotient map* $\mathcal{U}_\mu^- \to \mathcal{U}_0^-/\mathcal{U}^{-,\mu}$ *is an isomorphism.*

Identical assertions hold with μ *anti-dominant and each* $\mathcal{U}_0^-$ *replaced by* $\mathcal{U}_0^+ = \mathcal{G}_0 \cap U^+((z^{-1}))$*.*

Proof. For part (1) one shows that the orthogonal complement of $\operatorname{Lie}\mathcal{U}^{-,\mu}$ in $\mathfrak{g}[z]$ is a Lie subalgebra, which is an easy computation. Part (2) follows since multiplication $\mathcal{U}_\mu^+ \times \mathcal{U}^{+,\mu} \to \mathcal{U}_0^+$ is an isomorphism. □

Recall $\mathrm{Gr}_\mu \subset \mathrm{Gr}^{\mathrm{thick}}$ is the $\mathcal{G}_0$-orbit through z^μ for dominant $\mu \in X_*(T)$. Lemma 3.2 furnishes two descriptions of this locally closed subscheme. Firstly, the orbit map induces an isomorphism $\mathcal{G}_0/\mathcal{U}^{-,\mu} \xrightarrow{\sim} \mathrm{Gr}_\mu$ which, by part (1) of Lemma 3.2, endows Gr_μ with a natural Poisson structure. On the other hand, part (2) of Lemma 3.2 combined with the factorisation (2.1) shows how acting on the base point in $\mathrm{Gr}^{\mathrm{thick}}$ gives an isomorphism $\mathcal{W}_\mu := \mathcal{U}_0^+ \mathcal{T}_0 u^\mu \mathcal{U}^- \xrightarrow{\sim} \mathrm{Gr}_\mu$.

In what follows it will also be useful to consider a variant of this construction giving an isomorphism

$$\mathcal{U}^{+,-\mu}\backslash\mathcal{G}_0/\mathcal{U}^{-,\mu} \to \mathcal{U}_{-\mu}^+ \times \mathcal{T}_0 \times \mathcal{U}_\mu^- \xrightarrow{g \mapsto z^\mu g z^\mu} \mathcal{W}_{2\mu} \tag{3.1}$$

where the first map comes from part (2) of Lemma 3.2. We claim (3.1) is Poisson for the quotient Poisson structure on the source and that on the target induced by the isomorphism $\mathcal{W}_{2\mu} \cong \mathrm{Gr}_{2\mu}$. To see this note the composite $\mathcal{G}_0 \to \mathcal{U}^{+,-\mu}\backslash\mathcal{G}_0/\mathcal{U}^{-,\mu} \to \mathcal{W}_{2\mu}$ is exactly the shift morphism denoted $\iota_{2\mu,-\mu,-\mu}$ in [FKP+18, §5.9]. That this is Poisson follows from [FKP+18, Theorem 5.15].

3.2. **Symmetric quotients and fixed points.** Consider the anti-involution $\tau(g) = g^t(-z)$ on $G((z^{-1}))$ and set

$$K = \{g \in G((z^{-1})) \mid \tau(g) = g^{-1}\}$$

This can be interpreted as the loop group associated to the special unitary group over $\mathbb{C}((z^{-1}))$ respecting the Hermitian form $(x, y) = x^t(z)y(-z)$ on $\mathbb{C}((z^{-1}))^n$. Thus, [PR08, Theorem 0.1] ensures K is connected. The group K acts on $\mathrm{Gr}^{\mathrm{thick}}$ via left multiplication.

Remark 3.3. Previous work [Nad04, CY23] consider the action on $\mathrm{Gr}^{\mathrm{thick}}$ of fixed points in $G((z^{-1}))$ under involutions like $g \mapsto g^{-t}$ on Gr. While there are a number of similarities, the geometry of this action is different to ours in several significant ways.

We avoid discussion of the quotient stack $K\backslash\,\mathrm{Gr}^{\mathrm{thick}}$ and instead restrict attention to $K_0\backslash\,\mathrm{Gr}_\mu$ for $K_0 = K \cap \mathcal{G}_0$. The latter are represented by schemes, since $g \mapsto \tau(g)g$

induces a monomorphism

$$\Psi : K_0 \backslash \operatorname{Gr}_\mu = K_0 \backslash \mathcal{G}_0 / \mathcal{U}^{-,\mu} \to \mathcal{U}^{+,-\mu} \backslash \mathcal{G}_0 / \mathcal{U}^{-,\mu} \xrightarrow{(3.1)} \mathcal{W}_{2\mu}^{\tau=1} \tag{3.2}$$

which is easily checked, e.g. by comparing Hilbert series of the tangent spaces at the identity, graded via loop rotations, to be surjective and hence an isomorphism.

Lemma 3.4. *The subgroup $K_0 \subset \mathcal{G}_0$ is coisotropic.*

Proof. As in Lemma 3.2, this follows since the orthogonal complement of Lie K_0 in $\mathfrak{g}[z]$ is Lie$(K \cap G[z])$ which is a Lie subaglebra. ☐

Thus $K_0 \backslash \operatorname{Gr}_\mu = K_0 \backslash \mathcal{G}_0 / \mathcal{U}^{-,\mu}$ has a natural Poisson structure and (3.2) transfers this structure to $\mathcal{W}_{2\mu}^{\tau=1}$. On the other hand, $(\mathcal{U}^{+,-\mu} \backslash \mathcal{G}_0 / \mathcal{U}^{-,\mu})^{\tau=1} \cong \mathcal{W}_{2\mu}^{\tau=1}$ has an intrinsic Poisson structure induced from that on $\operatorname{Gr}_{2\mu} \cong \mathcal{W}_{2\mu}$ via Dirac reduction (see, for example, [LGPV13, §5.4.3], [Top23, §2]). As explained in [LGPV13, Proposition 5.36], the corresponding Poisson bracket $\{-,-\}_\tau$ is given by

$$\{F, G\}_\tau = \frac{1}{2} \left(\{\widetilde{F}, \widetilde{G}\} + \{\tau^* \widetilde{F}, \widetilde{G}\} \right) \tag{3.3}$$

where $\{-,-\}$ denotes the Poisson bracket on $\mathcal{W}_{2\mu}$ and $\widetilde{F}, \widetilde{G} \in \mathcal{O}(\mathcal{W}_{2\mu})$ are lifts of $F, G \in \mathcal{O}(\mathcal{W}_{2\mu}^{\tau=1})$. In fact, these two Poisson structures on $\mathcal{W}_{2\mu}^{\tau=1}$ coincide up to a multiple of 2, as follows from [Xu03, Theorem 5.9]. More precisely, one applies loc. cit. (which, while written in a finite dimensional setting, goes through immediately in our loop setup) in the case $\mu = 0$, and then deduces the claim for $\mu > 0$ using functoriality of Dirac reduction for fixed points as described in [Top23, §2].

3.3. Symplectic leaves and their closures.

Definition 3.5. *For dominant $\lambda \geq 2\mu$ write*

$$\mathcal{S}_{2\mu}^{\lambda} := \Psi^{-1}(\mathcal{W}_{2\mu}^{\tau=1} \cap G[z] z^\lambda G[z])$$

for Ψ as in (3.2) and consider the closed subset $\mathcal{S}_{2\mu}^{\leq \lambda} = \bigcup_{\gamma \leq \lambda} \mathcal{S}_{2\mu}^{\gamma}$ equipped with the reduced scheme structure.

Recall from [KWWY14, Theorem 2.5] that the intersections $\mathcal{W}_{2\mu} \cap G[z] z^\lambda G[z]$ are symplectic leaves inside $\mathcal{W}_{2\mu}$. These exhaust all symplectic leaves meeting $G[z, z^{-1}]$. It follows from [LGPV13, Proposition 5.26] that the connected components of $\mathcal{W}_{2\mu}^{\tau=1} \cap G[z] z^\lambda G[z]$ are then the symplectic leaves in $\mathcal{W}_{2\mu}^{\tau=1}$. The same is therefore true of the $\mathcal{S}_{2\mu}^{\lambda}$.

Proposition 3.6. *If $\mathcal{S}_{2\mu}^{\lambda} \neq \emptyset$ then $\lambda = \eta + w(\eta)$ for some $\eta \in X_*(T)^+$ and some involution $w \in W$ (i.e. $w^2 = 1$). If one can take w with at least one fixed point then*

$$\mathcal{S}_{2\mu}^{\lambda} = K_0 \backslash (K x u^\eta \cap \operatorname{Gr}_\mu)$$

for any $x \in G$ with $\tau(x)x \in N(T)$ lifting w. Otherwise there are $x^\pm \in G$ with $\tau(x^\pm)x^\pm \in N(T)$ lifting w so that $K_0\backslash(Kx^\pm u^\eta \cap \mathrm{Gr}_\mu)$ are distinct—in this case $\mathcal{S}^\lambda_{2\mu}$ is the disjoint union of these two intersections.

Note that if $\lambda = \sum_{1\le i\le n}\lambda_i\epsilon_i$ then $\lambda = \eta + w(\eta)$ as in the Proposition 3.6 if and only if the number of λ_i equal any given odd number is even. The second case of Proposition 3.6 occurs when every λ_i is odd (and is therefore only possible when n is even and ≥ 4).

Proof. Consideration of the map (3.2) shows that $K_0 q \in K_0\backslash \mathrm{Gr}_\mu$ lies inside $\mathcal{S}^\lambda_{2\mu}$ if and only if $\tau(q_0)q_0 \in G[z]z^\lambda G[z]$ for any $q_0 \in G((z^{-1}))$ representing $q \in \mathrm{Gr}_\mu$. Notice this implies $K_0 q \cap \mathrm{Gr}^{\mathrm{thin}} \neq \emptyset$.

On the other hand, [DS03, Theorem 5.2] shows that any $q \in \mathrm{Gr}^{\mathrm{thin}}$ can be represented by $q_0 \in G[z,z^{-1}]$ with $\tau(q_0)q_0 = \dot{w}z^\lambda$ where

- $\lambda = (\lambda_1,\ldots,\lambda_n) \in X_*(T)^+$,
- $\dot{w} \in N(T)\cap G^\tau$ represents an involution $w \in W$ with $w(\lambda)=\lambda$ and whose fixed points are exactly the $i \in \{1,\ldots,n\}$ with $\lambda_i \in 2\mathbb{Z}$ and with $w(\lambda)=\lambda$.

This immediately implies $\lambda = \eta + w(\eta)$ for some $\eta \in X_*(T)^+$. Now every element in $N(T)\cap G^\tau$ can be expressed as $\tau(x)x$ for some $x \in G$. Applying this to $(-1)^{\eta_\lambda}\dot{w}$ allows us to write $\dot{w}z^\lambda = \tau(xz^\eta)xz^\eta$. We conclude that the K-orbit through q equals Kxz^η. Consequently,

$$\mathcal{S}^\lambda_{2\mu} = K_0\backslash\Big(\bigcup Kxu^\eta \cap \mathrm{Gr}_\mu\Big)$$

with the union running over $x \in G$ with $\tau(x)x \in N(T)$ lifting w. It only remains to determine when two such Kxu^η coincide, and this reduces to a description of when the $K\cap G$-orbits through the class of x inside G/P_η coincide for $P_\eta \subset G$ the parabolic subgroup stabilising $u^\eta \in \mathrm{Gr}^{\mathrm{thick}}$. Such a description is given in [RS90, Lemma 10.3.1] and the proposition follows. □

Lemma 3.7. *If $\mu = 0$ then each subscheme $K_0\backslash(Kxz^\eta \cap \mathrm{Gr}_\mu)$ as in Proposition 3.6 is connected, and hence a symplectic leaf in $K_0\backslash \mathrm{Gr}_\mu$.*

Proof. It suffices to show $Kxz^\eta \cap \mathrm{Gr}_\mu$ is connected. If it is empty then we are done so assume not. As Gr_0 is open in $\mathrm{Gr}^{\mathrm{thick}}$ this intersection is open inside the K-orbit through xz^η. But K is a connected group and so this orbit is irreducible. The same must then be true of any non-empty open subset. □

For later use we record the following simple combinatorial consequence of the constraint on λ given in Proposition 3.6.

Corollary 3.8. *Suppose $\lambda = \sum_{1\le i\le n}\lambda_i\epsilon_i$ can be expressed as $\eta + w(\eta)$ for an involution $w \in W$. If*

$$r_i := \langle \omega_i, -w_0(\lambda)\rangle = -(\lambda_{n-i+1} + \ldots + \lambda_n)$$

then r_i odd implies $\lambda_{n-i} = \lambda_{n-i+1}$ are both odd, and so r_{i+1} is even.

In general, we do not know whether $\mathcal{S}_{2\mu}^{\leq\lambda}$ coincides with the closure of $\mathcal{S}_{2\mu}^{\lambda}$ in $K_0\backslash \mathrm{Gr}_\mu$ whenever $\mathcal{S}_{2\mu}^{\lambda}$ is non-empty. However, we are able to show this holds in the simplest case, which is when λ is even, i.e. lies in $2X_*(T)^+$.

Theorem 3.9. *If $\lambda \in 2X_*(T)^+$ then $\mathcal{S}_{2\mu}^{\leq\lambda}$ equals the closure of $\mathcal{S}_{2\mu}^{\lambda}$ in $K_0\backslash \mathrm{Gr}_\mu$ and coincides with the scheme theoretic image of $\mathrm{Gr}_\mu \cap \mathrm{Gr}^{\leq\eta} \to K_0\backslash \mathrm{Gr}_\mu$ (recall the notation from Section 2.2).*

Proof. Write $\overline{\mathcal{S}}_{2\mu}^{\lambda}$ for the closure of $\mathcal{S}_{2\mu}^{\lambda}$ in $K_0\backslash \mathrm{Gr}_\mu$ and set $\lambda = 2\eta$. We then consider the K^0-orbit $\mathcal{O}^0$ through u^η where $K^0 := K \cap G[z]$. We first claim that the closure of $\mathcal{O}^0$ in $\mathrm{Gr}^{\mathrm{thick}}$ equals $\mathrm{Gr}^{\leq\eta}$. Certainly this closure is contained in $\mathrm{Gr}^{\leq\eta}$ and, since the latter is irreducible, equality follows if $\mathcal{O}^0$ and $\mathrm{Gr}^{\leq\eta}$ have the same dimension, i.e. if $\dim \mathcal{O}^0 = \sum_{\alpha^\vee>0}\langle\alpha^\vee,\eta\rangle$. Since the orbit map identifies $\mathcal{O}^0 \cong K^0/(K^0 \cap z^\eta G[z]z^{-\eta})$ this dimension can be computed as

$$\dim \mathrm{Lie}\, K^0/\left(\mathrm{Lie}\, K^0 \cap z^\eta \mathfrak{g}[z]z^{-\eta}\right)$$

But $\mathrm{Lie}\, K^0 = \{x \in \mathfrak{g}[z] \mid \tau(x) = -x\}$ is spanned by $z\,\mathrm{Lie}\, T[z^2]$ together with the elements $z^i(x_{\alpha^\vee} - (-1)^r x_{-\alpha^\vee})$ for all $i \geq 0$ and $\alpha^\vee > 0$. As η is dominant it follows that the above quotient is spanned by the images of $z^i(x_{\alpha^\vee} - (-1)^r x_{-\alpha^\vee})$ for $0 \leq i < \langle\alpha^\vee,\eta\rangle$ and $\alpha^\vee > 0$. This proves the claim.

Next we show that $\mathrm{Gr}^{\leq\lambda} \cap \mathrm{Gr}_\mu$ equals the closure in Gr_μ of $\mathcal{O}^0 \cap \mathrm{Gr}_\mu$. Since $\mathcal{O}^0$ is dense in $\mathrm{Gr}^{\leq\eta}$ by the previous paragraph and $z^\mu \mathrm{Gr}_0$ is open in $\mathrm{Gr}^{\mathrm{thick}}$ with $z^\mu \mathrm{Gr}_0 \cap \mathrm{Gr}^{\leq\eta}$ non-empty, we deduce that $\mathcal{O}^0 \cap z^\mu \mathrm{Gr}_0$ is non-empty and has closure $\mathrm{Gr}^{\leq\eta} \cap \mathrm{Gr}_\mu$ inside Gr_μ. To replace $z^\mu \mathrm{Gr}_0$ with Gr_μ in this assertion notice that

$$z^\mu \mathrm{Gr}_0 \cong z^\mu \mathcal{U}_0^+ z^{-\mu}\mathcal{T}_0 z^\mu \mathcal{U}_0^- \cong \left(z^\mu \mathcal{U}_0^+ z^{-\mu} \cap G[z]\right) \times \mathcal{W}_\mu$$

where the second isomorphism uses Lemma 3.2 to identify $\mathcal{U}_0^+ = \mathcal{U}^{+,-\mu} \times \mathcal{U}_{-\mu}^+$, and so identify $z^\mu \mathcal{U}_0^+ z^{-\mu} = \left(z^\mu \mathcal{U}_0^+ z^{-\mu} \cap G[z]\right) \times \mathcal{U}_0^+$. Let $p : z^\mu \mathrm{Gr}_0 \to \mathrm{Gr}_\mu$ be the resulting projection. Since $\mathrm{Gr}^{\leq\eta}$ is $G[z]$-stable we have $p^{-1}(\mathrm{Gr}^{\leq\eta} \cap \mathrm{Gr}_\mu) = \mathrm{Gr}^{\leq\eta} \cap z^\mu \mathrm{Gr}_0$. Now suppose $Z \subset \mathrm{Gr}^{\leq\eta} \cap \mathrm{Gr}_\mu$ is closed and contains $\mathcal{O}^0 \cap \mathrm{Gr}_\mu$. By the first assertion of this paragraph it follows that $p^{-1}(Z) = \mathrm{Gr}^{\leq\eta} \cap z^\mu \mathrm{Gr}_0$ and so $Z = \overline{\mathcal{O}}^0 \cap \mathrm{Gr}_\mu$. This shows $\mathrm{Gr}^{\leq\eta} \cap \mathrm{Gr}_\mu$ is the closure of $\mathcal{O}^0 \cap \mathrm{Gr}_\mu$ as required.

Proposition 3.6 shows $\mathcal{S}_{2\mu}^{\lambda} = K_0\backslash(Kz^\eta \cap \mathrm{Gr}_\mu)$ when $\lambda = 2\eta$. The well-known fact that multiplication $\mathcal{G}_0 \times G[z] \to G((z^{-1}))$ is an open immersion implies the same for $K_0 \times K^0 \to K$. As a consequence, the image of $\mathcal{O}^0 \cap \mathrm{Gr}_\mu \to K_0\backslash \mathrm{Gr}_\mu$ is dense inside $\mathcal{S}_{2\mu}^{\lambda}$. The previous paragraph therefore shows that $\mathrm{Gr}^{\leq\lambda} \cap \mathrm{Gr}_\mu \to K_0\backslash \mathrm{Gr}_\mu$ factors through $\overline{\mathcal{S}}_{2\mu}^{\lambda}$ with dense image. It follows that $\overline{\mathcal{S}}_{2\mu}^{\lambda}$ is the scheme theoretic image of this morphism.

Now suppose $\gamma \leq \lambda$ with $\mathcal{S}_{2\mu}^{\gamma}$ non-empty. Combining Proposition 3.6 with the openness of multiplication $K_0 \times K^0 \to K$ shows that, as ν runs over coweights with

$\gamma = \nu + w(\nu)$ with $w \in W$ an involution represented by $\tau(g)g$ for $g \in G$, the images of $K^0 x u^\nu \cap \mathrm{Gr}_\mu$ in $K_0 \backslash \mathrm{Gr}_\mu$ are each dense in a union of connected components of $\mathcal{S}^\gamma_{2\mu}$. Since $\gamma \leq \lambda$, we have $\nu \leq \eta$. Thus, any such $K^0 x u^\nu \cap \mathrm{Gr}_\mu$ is contained inside $\mathrm{Gr}^{\leq \eta} \cap \mathrm{Gr}_\mu$. We conclude that $\mathcal{S}^\gamma_{2\mu} \subset \overline{\mathcal{S}}^\lambda_{2\mu}$. This finishes the proof since clearly $\overline{\mathcal{S}}^\lambda_{2\mu} \subset \bigcup_{\gamma \leq \lambda} \mathcal{S}^\gamma_{2\mu}$. $\square$

For the remaining $\lambda \geq 2\mu$, i.e. those not in $2X_*(T)^+$ but of the form $\eta + w(\eta)$ for an involution w, we expect that $\mathcal{S}^\lambda_{2\mu}$ is non-empty. Indeed, computations made when $\mu = 0$ suggest that $\mathcal{S}^\lambda_{2\mu}$ is irreducible (except in those cases described in Proposition 3.6) of dimension

$$\sum_{1 \leq i \leq n-1} 2\lfloor \frac{n_i}{2} \rfloor$$

for n_i defined by $\lambda = 2\mu + \sum_{1 \leq i \leq n-1} n_i \alpha_i$.

3.4. **Ideal generators.** Our goal in this section is to describe Poisson generators of the ideal of $\mathcal{S}^{\leq \lambda}_{2\mu}$. We are currently only able to do this when $\mu = 0$ and we impose this restriction throughout this section. In particular, $\Psi : K_0 \backslash \mathcal{G}_0 \to \mathcal{G}_0^{\tau = 1}$ from (3.2) is simply the map $K_0 x_0 \mapsto \tau(x_0) x_0$.

Notation 3.10. For i-tuples $I, J \subset \{1, \ldots, n\}$ recall $\Delta_{IJ} \in \mathcal{O}(\mathcal{G}_0)$ is defined in Section 2.2. Set

$$\Delta^\tau_{IJ} := \Delta_{IJ} \circ \Psi \in \mathcal{O}(K_0 \backslash \mathcal{G}_0)$$

and $\Delta^{\tau,(r)}_{IJ}$ for the coefficient of z^{-r} in this series. Thus $\Delta^\tau_{IJ}(K_0 x_0) = \Delta_{IJ}(\tau(x)x)$.

Of particular relevance will be the functions

$$A^{(r)}_i := \Delta^{\tau,(r)}_{\{n-i+1,\ldots,n\},\{n-i+1,\ldots,n\}}, \qquad B^{(r)}_i := \Delta^{\tau,(r)}_{\{n-i+1,\ldots,n\},\{n-i,n-i+2,\ldots,n\}}$$

lying inside $\mathcal{O}(K_0 \backslash \mathcal{G}_0)[[z^{-1}]]$.

Proposition 3.11. *Suppose $\mathcal{S} \subset \mathcal{S}^\lambda_0$ is a symplectic leaf with $\lambda = \sum_{1 \leq i \leq n} \lambda_i \epsilon_i \in X_*(T)^+$ not necessarily in $2X_*(T)^+$ and set*

$$r_i = \langle \omega_i, -w_0 \lambda \rangle = -(\lambda_n + \ldots + \lambda_{n-i+1})$$

Then, as functions on $\mathcal{S}$, $A^{(r_{i+1})}_{i+1} \neq 0$ if r_{i+1} is even, and $B^{(r_{i+1})}_{i+1} \neq 0$ if r_{i+1} is odd.

Before giving the proof recall that if $g \in G[z] z^\lambda G[z]$ then, for each pair of i-tuples $I, J \subset \{1, \ldots, n\}$, the minor $\Delta_{IJ}(g)$ has z-adic valuation $\geq \langle \omega_i, -w_0(\lambda) \rangle$. Furthermore, for each $1 \leq i \leq n$ there is an I, J for which this is an equality. In the proof of Proposition 3.11 we will need the following more specific version of this assertion:

Lemma 3.12. *Let $\lambda = \sum_{1 \leq i \leq n} \lambda_i \epsilon_i$ and suppose $g \in G[z] z^\lambda G[z]$ is such that $\Delta_{\{n-i+1,\ldots,n\},\{n-i+1,\ldots,n\}}(g)$ has z-adic valuation $\lambda_n + \ldots + \lambda_{n-i+1}$. Then there are $1 \leq r, s \leq n - i$ for which*

$$\Delta_{\{r,n-i+1,\ldots,n\},\{s,n-i+1,\ldots,n\}}(g)$$

has z-adic valuation $\lambda_n + \ldots + \lambda_{n-i}$.

Proof. The hypothesis on g says that if $g = \left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)$ with W an $i-1$ by $i-1$ matrix then $\det(W)$ has z-adic valuation $\lambda_n + \ldots + \lambda_{n-i+1}$. Since the maximal minors of Z and Y have z-adic valuation at least that of $\det W$ we can find matrices Y^*, Z^* so that

$$g' := \begin{pmatrix} 1 & Y^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Z^* & 1 \end{pmatrix} = \begin{pmatrix} X' & 0 \\ 0 & W \end{pmatrix}$$

Note $\Delta_{\{r,n-i+1,\ldots,n\},\{s,n-i+1,\ldots,n\}}(g') = \Delta_{\{r,n-i+1,\ldots,n\},\{s,n-i+1,\ldots,n\}}(g)$ for any $1 \le r, s \le n-i$, as can be seen by considering the action of $\left(\begin{smallmatrix} 1 & Z^* \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ Y^* & 1 \end{smallmatrix}\right)$ on the relevant element inside $\bigwedge^i \mathbb{C}^n$. On the other hand, the fact that $\left(\begin{smallmatrix} X' & 0 \\ 0 & W \end{smallmatrix}\right) \in G[z]z^\lambda G[z]$ ensures $\Delta_{\{r,n-i+1,\ldots,n\},\{s,n-i+1,\ldots,n\}}(g')$ has z-adic valuation $\lambda_n + \ldots + \lambda_{n-i}$ for some $1 \le r, s \le n-i$. □

Proof of Proposition 3.11. Argue by induction on $i+1$. If r_i is even (or $i = 0$) then we can assume the existence of $K_0 x_0 \in K_0 \backslash \mathcal{G}_0$ with $A_i^{(r_i)}(K_0 x_0) \neq 0$. Applying Lemma 3.12 therefore produces $1 \le r, s \le n-i$ with

$$\Delta^{\tau,(r_{i+1})}_{\{r,n-i+1,\ldots,n\},\{s,n-i+1,\ldots,n\}}$$

non-vanishing on $K_0 x_0$. To prove the proposition in this case we will vary the point $K_0 x_0$ inside $\mathcal{S}$ by flowing along suitable Hamiltonian vector fields.

More precisely, we will use the following observation: If $f \in \mathcal{O}(K_0 \backslash \mathcal{G}_0)$ and $x \in \mathcal{S}_0^\lambda$ is a closed point with

$$\{f, g\}_\tau(x) \neq 0$$

for some $g \in \mathcal{O}(K_0 \backslash \mathcal{G}_0)$ then there exists a closed point $y \in \mathcal{S}_0^\lambda$ with $f(y) \neq 0$. Indeed, $\mathcal{S}_0^\lambda$ being a symplectic leaf means that the tangent space $T_x \mathcal{S}_0^\lambda$ is spanned by the value at x of the Hamiltonian vector fields on $K_0 \backslash \mathcal{G}_0$, while $\{f, g\}_\tau(x)$ is by definition the value of f on the Hamiltonian vector field associated to g at x. Since $\mathcal{S}_0^\lambda$ is smooth, and hence reduced, it follows that the vanishing locus of f has codimension 1 in $\mathcal{S}_0^\lambda$.

Step 1. First, we claim x_0 can be chosen so that at least one of r and s equals 1. Suppose for a contradiction that this is not the case. Then (3.3) and Lemma 3.1 together give

$$\begin{aligned} \{\Delta^{\tau,(r_{i+1})}_{\{n-i,n-i+1,\ldots,n\},\{s,n-i+1,\ldots,n\}}, \Delta^{\tau,(1)}_{n-i,r}\}_\tau = \\ &= \frac{1}{2}\{\Delta^{(r_{i+1})}_{\{n-i,n-i+1,\ldots,n\},\{s,n-i+1,\ldots,n\}}, \Delta^{(1)}_{n-i,r}\} \circ \Psi \\ &\quad - \frac{1}{2}\{\Delta^{(r_{i+1})}_{\{n-i,n-i+1,\ldots,n\},\{s,n-i+1,\ldots,n\}}, \Delta^{(1)}_{r,n-i}\} \circ \Psi \\ &= \frac{1}{2}\Delta^{\tau,(r_{i+1})}_{\{r,n-i+1,\ldots,n\},\{s,n-i+1,\ldots,n\}} \end{aligned}$$

as functions on $\mathcal{S}$ (we remind the reader here that $\{-,-\}_\tau$ denotes the Poisson bracket on $K_0\backslash\mathcal{G}_0$ pulled back via Ψ from that on $\mathcal{G}_0^{\tau=1}$ obtained via Dirac reduction from the bracket $\{-,-\}$ on $\mathcal{G}_0$). But by assumption $\Delta^{\tau,(r_{i+1})}_{\{r,n-i+1,\dots,n\},\{s,n-i+1,\dots,n\}}$ does not vanish on $\mathcal{S}$ and so the argument of the previous paragraph produces a point in $\mathcal{S}$ on which $\Delta^{\tau,(r_{i+1})}_{\{n-i,n-i+1,\dots,n\},\{s,n-i+1,\dots,n\}}$ does not vanish.

Step 2. If r_{i+1} is even then

$$\tau^*\Delta^{(r_{i+1})}_{\{n-i,\dots,n\},\{n-i,\dots,n\}} = \Delta^{(r_{i+1})}_{\{n-i,\dots,n\},\{n-i,n-i+1,\dots,n\}}$$

inside $\mathcal{O}(\mathcal{G}_0)$. This, together with (3.3) and Lemma 3.1 combine to give

$$\begin{aligned}\{\Delta^{\tau,(r_{i+1})}_{\{n-i,\dots,n\},\{n-i,\dots,n\}}, \Delta^{\tau,(1)}_{s,n-i}\}_\tau &= \{\Delta^{(r_{i+1})}_{\{n-i,\dots,n\},\{n-i,n-i+1,\dots,n\}}, \Delta^{(1)}_{s,n-i}\} \circ \Psi \\ &= \Delta^{(\tau,r_{i+1})}_{\{n-i,\dots,n\},\{s,n-i+1,\dots,n\}}\end{aligned}$$

as functions on $\mathcal{S}$. By Step 1 we know this function is non-vanishing, and so we produce a point of $\mathcal{S}$ on which $\Delta^{\tau,(r_{i+1})}_{\{n-i,n-i+1,\dots,n\},\{n-i,n-i+1,\dots,n\}}$ does not vanish.

Step 3. If instead r_{i+1} is odd then we can assume $s \neq n-i-1$ (since otherwise we are done) and $s \neq n-i$ (since then $\Delta^{\tau,(r_{i+1})}_{\{n-i,n-i+1,\dots,n\},\{s,n-i+1,\dots,n\}} = 0$). Using (3.3) together with Lemma 3.1 then gives the identity

$$\begin{aligned}&\{\Delta^{\tau.(r_{i+1})}_{\{n-i,\dots,n\},\{n-i-1,n-i+1,\dots,n\}}, \Delta^{(1)}_{s,n-i-1}\}_\tau \\ &= \{\Delta^{(r_{i+1})}_{\{n-i,\dots,n\},\{n-i-1,n-i+1,\dots,n\}}, \Delta^{(1)}_{s,n-i-1}\} \circ \Psi - \{\Delta^{(r_{i+1})}_{\{n-i,\dots,n\},\{n-i-1,n-i+1,\dots,n\}}, \Delta^{(1)}_{n-i-1,s}\} \circ \Psi \\ &= -\Delta^{\tau,(r_{i+1})}_{\{n-i,\dots,n\},\{s,n-i+1,\dots,n\}}\end{aligned}$$

Again, by Step 1 we know this function is non-vanishing, and so we produce a point of $\mathcal{S}$ on which $\Delta^{\tau,(r_{i+1})}_{\{n-i,n-i+1,\dots,n\},\{n-i-1,n-i+1,\dots,n\}}$ does not vanish.

Step 4. We conclude with the case r_i is odd. By assumption $\mathcal{S}_0^\lambda$ is non-empty and so Corollary 3.8 forces r_{i+1} to be even and $\lambda_{n-i+1} = \lambda_{n-i}$ to be odd. The inductive hypothesis gives $K_0x \in K_0\backslash\mathcal{G}_0$ for which the series valued function

$$\Delta^\tau_{\{n-i+1,\dots,n\},\{n-i,n-i+2,\dots,n\}}$$

has z-adic valuation r_i. On the other hand, the Desnanot–Jacobi formula [VV23] gives

$$\Delta^\tau_{\{n-i,\dots,n\},\{n-i,\dots,n\}}\Delta^\tau_{\{n-i+2,\dots,n\},\{n-i+2,\dots,n\}} = \alpha - \beta \tag{3.4}$$

for

$$\begin{aligned}\alpha &= \Delta^\tau_{\{n-i+1,\dots,n\},\{n-i+1,\dots,n\}}\Delta^\tau_{\{n-i,n-i+2,\dots,n\},\{n-i,n-i+2,\dots,n\}}, \\ \beta &= \Delta^\tau_{\{n-i,n-i+2,\dots,n\},\{n-i+1,\dots,n\}}\Delta^\tau_{\{n-i+1,\dots,n\},\{n-i,n-i+2,\dots,n\}}\end{aligned}$$

A priori the two minors in α have z-adic valuation $\geq r_i$. In fact this inequality is strict because r_i is odd and so τ-invariance forces the coefficient of z^{-r_i} to vanish. Therefore, the right hand side of (3.4) has z-adic valuation $2r_i$. But recall $\lambda_{n-i+1} = \lambda_{n-i}$ and so $2r_i = r_{i+1} + r_{i-1}$. We conclude that $\Delta^{\tau}_{\{n-i,\dots,n\},\{n-i,\dots,n\}}$ and $\Delta^{\tau}_{\{n-i+2,\dots,n\},\{n-i+2,\dots,n\}}$, which respectively have z-adic valuations $\geq r_{i+1}$ and $\geq r_{i-1}$, in fact have exact z-adic valuations r_{i+1} and r_{i-1}. In particular, $A^{(r_{i+1})}_{i+1} \neq 0$ on $\mathcal{S}$ and we are done. $\square$

Corollary 3.13. *Assume $\lambda \in X_*(T)$ can be written as $\eta + w(\eta)$ for an involution $w \in W$. Then the ideal defining $\mathcal{S}^{\leq\lambda}_0$ inside $\mathcal{O}(K_0\backslash\mathcal{G}_0)$ is the radical of the ideal Poisson generated by the functions*

- *$A^{(r)}_i$ for $1 \leq i \leq n-1$ and $r > \langle \omega_i, -w_0\lambda\rangle$.*

Proof. Write J^λ_0 for ideal Poisson generated by the elements described, and $V(J^\lambda_0) \subset K_0\backslash \mathrm{Gr}_0$ for its vanishing locus. Clearly $\mathcal{S}^{\leq\lambda}_0 \subset V(J^\lambda_0)$ because if I, J are i-tuples then $\Delta^{\tau,(r)}_{IJ}$ vanishes on $\mathcal{S}^\lambda_0$ whenever $r > \langle \omega_i, -w_0\lambda\rangle$.

For the opposite inclusion we claim that J^λ_0 contains all $\Delta^{\tau,(r)}_{ij}$ with $1 \leq i, j \leq n$ and sufficiently large r. This ensures that the image of $V(J^\lambda_0)$ under Ψ lies inside $\mathrm{Gr}^{\mathrm{thin}}$, and so that $V(J^\lambda_0)$ is a union of the symplectic leaves $\mathcal{S} \subset \mathcal{S}^\gamma_0$ for varying γ. Granting this, the corollary follows from Proposition 3.11. Indeed, if $\gamma = \sum_{1\leq i\leq n} \gamma_i\epsilon_i > \lambda = \sum_{1\leq i\leq n} \lambda_i\epsilon_i$ then there is a smallest i with $r'_i := \langle \omega_i, -w_0\delta\rangle > \langle \omega_i, -w_0\lambda\rangle := r_i$. If r'_i is even then Proposition 3.11 shows $A^{(r'_i)}_i \neq 0$ on $\mathcal{S}$ and so $\mathcal{S} \not\subset V(J^\lambda_0)$. If r'_i is odd then Corollary 3.8 implies $r'_{i+1} = r'_i - \gamma_{n-i} = r'_i - \gamma_{n-i+1}$ is even. On the other hand, since $r'_{i-1} = r_{i-1}$ and $r'_i > r_i$ we must have $-\gamma_{n-i+1} > -\lambda_{n-i+1}$. Thus, $r'_{i+1} > r_{i+1}$ and Proposition 3.11 shows $A^{(r'_{i+1})}_i \neq 0$ on $\mathcal{S}$, hence $\mathcal{S} \not\subset V(J^\lambda_0)$ and we are done.

It only remains to prove our claim. If $i, j \neq 1$ and r is even then Lemma 3.1 combined with (3.3) gives $\{\Delta^{\tau,(r)}_{1,1}, \Delta^{\tau,(1)}_{1,j}\}_\tau = \Delta^{\tau,(r)}_{1,j}$ and

$$\{\Delta^{\tau,(r)}_{1,j}, \Delta^{\tau,(1)}_{i,1}\}_\tau = \frac{1}{2}\{\Delta^{(r)}_{1,j}, \Delta^{(1)}_{i,1}\} \circ \Psi + \frac{1}{2}\{\Delta^{(r)}_{j,1}, \Delta^{(1)}_{i,1}\} \circ \Psi = \frac{1}{2}\left(\delta_{ij}\Delta^{\tau,(r)}_{11} - \Delta^{\tau,(r)}_{ij}\right)$$

This shows J^λ_0 contains $\Delta^{\tau,(r)}_{ij}$ for all even $r > r_1$. On the other hand, if r is even then (3.3) combined with Lemma 3.1 gives

$$\{\Delta^{\tau,(r)}_{ij}, \Delta^{\tau,(2)}_{ll}\}_\tau = \{\Delta^{(r)}_{ij}, \Delta^{(2)}_{ll}\} \circ \Psi = -\Delta^{\tau,(r+1)}_{il}\delta_{lj} + \Delta^{\tau,(r+1)}_{lj}\delta_{il} - \Delta^{\tau,(r)}_{il}\Delta^{\tau,(1)}_{lj} + \Delta^{\tau,(r)}_{lj}\Delta^{\tau,(1)}_{il}$$

We've already seen that the final two terms on the right lie in J^λ_0. Therefore $\Delta^{\tau,(r+1)}_{il}\delta_{lj} - \Delta^{\tau,(r+1)}_{lj}\delta_{il} \in J^\lambda_0$. It follows that $\Delta^{\tau,(r+1)}_{il} \in J^\lambda_0$ if $i \neq l$ and $r > r_1$ is even. Since $\Delta^{\tau,(r+1)}_{ii} = 0$ for r even this finishes the proof. $\square$

We emphasise that it is essential to consider the radical of the ideal in Corollary 3.13. For example, suppose $\lambda \in 2X_*(T)^+$. Then $B^{(r_1+1)}_1$ vanishes on $\mathcal{S}^\lambda_0$ but the ideal Poisson generated by the $A^{(r)}_i$'s for $r > r_i$ only consists of functions with loop grading $\geq r_1 + 2$ (since the bracket has degree -1 for this grading). We believe that this non-reducedness can be eliminated by adding the $B^{(r_i+1)}_i$'s to the Poisson generating set whenever r_i is even.

Conjecture 3.14. *The ideal of $\mathcal{O}(K_0\backslash\mathcal{G}_0)$ which is Poisson generated by the functions*

- $A_i^{(r)}$ *for* $1 \leq i \leq n-1$ *and* $r > \langle\omega_i, -w_0\lambda\rangle$,
- $B_i^{(r_i+1)}$ *whenever* $r_i = \langle\omega_i, -w_0(\lambda)\rangle$ *is even*

is reduced, and hence is the ideal defining $\mathcal{S}_0^{\leq\lambda}$ *inside* $K_0\backslash\mathcal{G}_0$.

4. Shifted Yangians

Below we recall the definitions and main properties of shifted Yangians for $\mathfrak{sl}_n$.

4.1. **Drinfeld presentation.**

Given $\mu \in X_*(T)^+$, let

$$H_i(u) = u^{\alpha_i^\vee(\mu)} + \hbar \sum_{r=-\alpha_i^\vee(\mu)}^{\infty} H_i^{(r)} u^{-r-1}.$$

We call $\underline{H}_i(u) = \sum_{r=0}^{\infty} H_i^{(r)} u^{-r-1}$ the *principal part* of $H_i(u)$ (and use analogous notation for principal parts of other series). Also let

$$E_i(u) = \hbar \sum_{r=0}^{\infty} E_i^{(r)} u^{-r-1}, \qquad F_i(u) = \hbar \sum_{r=0}^{\infty} F_i^{(r)} u^{-r-1}.$$

The *μ-shifted Yangian* $Y_{\mu,\hbar}$ associated to $\mathfrak{sl}_n$ is the $\mathbb{C}[\hbar]$-algebra generated by $H_i^{(r)}, E_i^{(s)}$, $F_i^{(s)}$ (with $i \in \mathbb{I}, r \geq -\alpha_i^\vee(\mu), s \in \mathbb{N}$), subject to the following defining relations:

$$[H_i(u), E_j(v)] = -\tfrac{1}{2}a_{ij}\hbar \frac{[H_i(u), E_j(u) - E_j(v)]_+}{u-v},$$
$$[H_i(u), F_j(v)] = \tfrac{1}{2}a_{ij}\hbar \frac{[H_i(u), F_j(u) - F_j(v)]_+}{u-v},$$
$$[E_i(u), F_j(v)] = \delta_{ij}\hbar \frac{\underline{H}_i(u) - \underline{H}_i(v)}{u-v},$$
$$[E_i(u), E_i(v)] = -\hbar \frac{(E_i(u) - E_i(v))^2}{u-v},$$
$$[F_i(u), F_i(v)] = \hbar \frac{(F_i(u) - F_i(v))^2}{u-v},$$

$$[E_i(u), E_j(v)] = \tfrac{1}{2}\hbar \frac{[E_i(u), E_j(u) - E_j(v)]_+}{u-v} - \frac{[E_i^{(0)}, E_j(u) - E_j(v)]}{u-v} \quad (|j-i| = 1),$$
$$[F_i(u), F_j(v)] = -\tfrac{1}{2}\hbar \frac{[F_i(u), F_j(u) - F_j(v)]_+}{u-v} - \frac{[F_i^{(0)}, F_j(u) - F_j(v)]}{u-v} \quad (|j-i| = 1),$$

$$\mathrm{Sym}_{u_1,u_2}[E_i(u_1), [E_i(u_2), E_j(v)]] = 0 \qquad (|j-i| = 1),$$
$$\mathrm{Sym}_{u_1,u_2}[E_i(u_1), [E_i(u_2), E_j(v)]] = 0 \qquad (|j-i| = 1),$$

$$[E_i(u), E_j(v)] = [F_i(u), F_j(v)] = 0 \qquad (a_{ij} = 0).$$

We denote the specialization to $\hbar = 1$, i.e., $Y_{\mu,\hbar}/(\hbar - 1)Y_{\mu,\hbar}$, as Y_μ. If $\mu = 0$, we also abbreviate $Y_\hbar = Y_{0,\hbar}$ and $Y = Y_0$.

4.2. **PBW theorem.** For each positive root $\beta^\vee = \alpha_j + \alpha_{j+1} + \cdots + \alpha_i$ $(j \leq i)$, and $r \geq 0$, set

$$E_{\beta^\vee}^{(r)} = [[\cdots[E_j^{(r)}, E_{j+1}^{(0)}], \cdots, E_{i-1}^{(0)}], E_i^{(0)}],$$
$$F_{\beta^\vee}^{(r)} = [F_i^{(0)}, [F_{i-1}^{(0)}, \cdots, [F_{j+1}^{(0)}, F_j^{(r)}] \cdots]].$$

We call elements $E_{\beta^\vee}^{(r)}, F_{\beta^\vee}^{(r)}$ and $H_i^{(s_i)}$, as $\beta^\vee$ ranges over Δ^+, i over $\mathbb{I}$, $r \geq 0$ and $s_i \geq -\alpha_i^\vee(\mu)$, PBW variables. Fix any total ordering on the set of PBW variables.

Theorem 4.1 ([FT19, Theorem 2.55])**.** *Ordered PBW monomials in the PBW variables form a basis of $Y_{\mu,\hbar}$ as a free $\mathbb{C}[\hbar]$-module, as well as a basis of Y_μ as a free $\mathbb{C}$-module.*

As a consequence of Theorem 4.1 (c.f. [FKP+18, Corollary 3.16]), given anti-dominant coweights μ_1, μ_2, the shift homomorphisms

$$\iota(\mu, \mu_1, \mu_2) : Y_\mu \to Y_{\mu+\mu_1+\mu_2} \tag{4.1}$$

defined by

$$H_i^{(r)} \mapsto H_i^{(r-\alpha_i^\vee(\mu_1+\mu_2))}, \qquad E_i^{(r)} \mapsto E_i^{(r-\alpha_i^\vee(\mu_1))}, \qquad F_i^{(r)} \mapsto F_i^{(r-\alpha_i^\vee(\mu_2))}.$$

are injective. In particular, if $\mu_1 + \mu_2 = -\mu$, then $\iota(\mu, \mu_1, \mu_2)$ yields an embedding $Y_\mu \hookrightarrow Y$.

4.3. **Canonical filtration.** For the purpose of quantizing slices in the affine Grassmannian, one requires a certain subalgebra of $Y_{\mu,\hbar}$. Namely, let $\mathbf{Y}_\mu$ be the $\mathbb{C}[\hbar]$-subalgebra of $Y_{\mu,\hbar}$ generated by

$$\{\hbar E_{\beta^\vee}^{(r)}\}_{\beta^\vee \in \Delta^+}^{r \geq 0} \cup \{\hbar F_{\beta^\vee}^{(r)}\}_{\beta^\vee \in \Delta^+}^{r \geq 0} \cup \{\hbar H_i^{(s_i)}\}_{i \in \mathbb{I}}^{s_i \geq -\alpha_i^\vee(\mu)}.$$

This subalgebra also admits an alternative description as a Rees algebra, which we now recall. By [FKP+18, §5.4], given a pair of coweights μ_1, μ_2 such that $\mu_1 + \mu_2 = \mu$, there is a filtration $F_{\mu_1,\mu_2}^\bullet$ on Y_μ determined by

$$\deg E_{\beta^\vee}^{(r)} = \beta^\vee(\mu_1) + r + 1, \qquad \deg F_{\beta^\vee}^{(r)} = \beta^\vee(\mu_2) + r + 1, \qquad \deg H_i^{(r)} = \alpha_i^\vee(\mu) + r + 1.$$

According to [FKP+18], this defines an algebra filtration, is independent of the choice of PBW variables and their ordering, and the corresponding Rees algebra $\mathrm{Rees}^{F_{\mu_1,\mu_2}^\bullet} Y_\mu$ is independent of the choice of coweights μ_1, μ_2.

Theorem 4.2 ([FT19, Theorem 2.57]). *For any $\mu \in X_*(T)^+$, there is a canonical $\mathbb{C}[\hbar]$-algebra isomorphism*

$$\mathbf{Y}_\mu \cong \operatorname{Rees}^{F^\bullet_{\mu_1,\mu_2}} Y_\mu.$$

Passing to the semi-classical limit, there are canonical isomorphisms

$$\mathbf{Y}_\mu/\hbar\mathbf{Y}_\mu \cong \operatorname{Rees}^{F^\bullet_{\mu_1,\mu_2}} Y_\mu/\hbar \operatorname{Rees}^{F^\bullet_{\mu_1,\mu_2}} Y_\mu \cong \operatorname{gr}^{F^\bullet_{\mu_1,\mu_2}} Y_\mu.$$

Moreover, by [FKP$^+$18, Proposition 5.7], $\operatorname{gr}^{F^\bullet_{\mu_1,\mu_2}} Y_\mu$ is a commutative algebra. It is also naturally a Poisson algebra, with the Poisson bracket given by

$$\{a, b\} = \hbar^{-1}[\hat{a}, \hat{b}] \mod \hbar\mathbf{Y}_\mu,$$

for any lifts $\hat{a}, \hat{b}$. Finally, we remark that the shift homomorphisms (4.1) induce monomorphisms of Poisson algebras

$$\mathbf{Y}_\mu/\hbar\mathbf{Y}_\mu \hookrightarrow \mathbf{Y}_{\mu+\mu_1+\mu_2}/\hbar\mathbf{Y}_{\mu+\mu_1+\mu_2}.$$

4.4. **Quantum duality.** When $\mu = 0$, the algebra $\mathbf{Y} = \mathbf{Y}_0$ can be identified with the Drinfeld–Gavarini dual of $Y_\hbar$. Let us first recall the general definition of this notion.

Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{C}$, and suppose that A is a deformation–quantization of the Hopf algebra $U(\mathfrak{a})$, i.e., A is a Hopf algebra over $\mathbb{C}[\hbar]$, and there is an isomorphism of Hopf algebras $A/\hbar A \cong U(\mathfrak{a})$. Let Δ and ϵ be the coproduct and counit of A, respectively. For $m \geq 0$, define inductively $\Delta^m \colon A \to A^{\otimes n}$ by

$$\Delta^0 = \epsilon, \qquad \Delta^1 = \mathrm{id}, \qquad \Delta^m = (\Delta \otimes \mathrm{id}^{\otimes(m-2)}) \circ \Delta^{m-1}.$$

Also define

$$\delta_m \colon A \to A^{\otimes m}, \qquad \delta_m = (\mathrm{id} - \epsilon)^{\otimes m} \circ \Delta^m.$$

The *Drinfeld–Gavarini dual* A' of A is the following sub-Hopf algebra:

$$A' = \{a \in A \mid \delta_m(a) \in \hbar^m A^{\otimes m} \text{ for all } m \in \mathbb{N}\}.$$

The importance of A' derives from the fact that, according to the quantum duality principle [Gav02, Theorem 1.6], A' is a deformation–quantization of the coordinate ring of an algebraic group $G_{\mathfrak{a}^*}$ associated to the dual Lie algebra $\mathfrak{a}^*$.

Let us now return to the specific setting of the Yangian $Y_\hbar$. It is a graded Hopf algebra, with $\deg(\hbar) = 1$ and $\deg(x^{(r)}) = r$ for $x = H_i, E_{\beta^\vee}, F_{\beta^\vee}$. The description of the Hopf algebra structure can be found in, e.g., [FT19, (A.16)–(A.17)]. The Yangian is a deformation–quantization of the universal enveloping algebra of the current Lie algebra, i.e., $Y_\hbar/\hbar Y_\hbar \cong U(\mathfrak{sl}_n[t])$. The latter is an isomorphism of graded Hopf algebras if $U(\mathfrak{sl}_n[t])$ is endowed with the loop grading.

Theorem 4.3 ([FT19, Corollary A.22]). *There is a canonical $\mathbb{C}[\hbar]$-algebra isomorphism*

$$Y_\hbar' \cong \mathbf{Y}.$$

4.5. **RTT presentation.** The *RTT Yangian* $\widetilde{Y}_\hbar^{\mathsf{rtt}}$ for $\mathfrak{gl}_n$ is the algebra generated by $t_{ij}^{(k)}$, where $k \geq 1$ and $1 \leq i, j \leq n$, subject to the relations

$$(u-v)[t_{ij}(u), t_{kl}(v)] = \hbar(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)),$$

where

$$t_{ij}(u) = \delta_{ij} + \hbar \sum_{r>0} t_{ij}^{(r)} u^{-r}.$$

Writing

$$T(u) = \sum_{i,j=1}^{n} e_{ij} \otimes t_{ij}(u), \qquad P = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji} \qquad R(u) = 1 - Pu^{-1},$$

where e_{ij} are the usual matrix units, these relations can be rewritten as

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v).$$

For any formal series $f(u) \in 1 + \frac{\hbar}{u}\mathbb{C}[\hbar][[u^{-1}]]$, the assignment

$$T(u) \mapsto f(u)T(u) \tag{4.2}$$

defines an algebra automorphism of $\widetilde{Y}_\hbar^{\mathsf{rtt}}$. The RTT Yangian $Y_\hbar^{\mathsf{rtt}}$ for $\mathfrak{sl}_n$ is the $\mathbb{C}[\hbar]$-subalgebra of $\widetilde{Y}_\hbar^{\mathsf{rtt}}$ consisting of all the elements fixed under all automorphisms (4.2).

Let $\widetilde{\mathbf{Y}}^{\mathsf{rtt}}$ be the $\mathbb{C}[\hbar]$-subalgebra of $\widetilde{Y}_\hbar^{\mathsf{rtt}}$ generated by $\{\hbar t_{ij}^{(r)}\}_{1\leq i,j\leq n}^{r\geq 1}$, and let $\mathbf{Y}^{\mathsf{rtt}} = \widetilde{\mathbf{Y}}^{\mathsf{rtt}} \cap Y_\hbar^{\mathsf{rtt}}$.

Theorem 4.4. *There is a $\mathbb{C}[\hbar]$-algebra isomorphism $Y_\hbar \cong Y_\hbar^{\mathsf{rtt}}$, which restricts to an isomorphism $\mathbf{Y} \cong \mathbf{Y}^{\mathsf{rtt}}$.*

Proof. The isomorphism is constructed by Gauss decomposition, see, e.g., [Mol07, §1.11]. For the latter statement, see [FT19, Proposition 2.34]. ☐

In light of Theorem 4.4, we will, from now on, just use the notations $Y_\hbar$ and $\mathbf{Y}$.

5. Shifted twisted Yangians

Below we recall the definition of shifted twisted Yangians of type AI (in the current presentation) and their main properties. While we are principally interested in the twisted Yangians associated to $\mathfrak{sl}_n$, we will also need their $\mathfrak{gl}_n$-version. The latter has the advantage that it is easier to verify that the $\mathfrak{gl}_n$-version of the relations are preserved by the twisted GKLO homomorphism defined in §7.

5.1. **Generators and relations.** The Drinfeld presentations of the twisted Yangians of split type associated to $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ (i.e., type AI) were first found in [LWZ23, Theorem 5.1, Theorem 5.3] via Gauss decomposition. A generalization to all simple split types, excluding G_2, was established in [LWZ25, §4]. Presentations in terms of generating currents, which are more convenient for our purposes, can be found in [LWZ23, Remark 4.12] and [LWZ25, §4.3]. Below, in the $\mathfrak{gl}_n$ case, we use a presentation with a slightly different choice of Cartan generators than in [LWZ23, Theorem 5.1], which can be found in [LPT+25, Proposition 5.3]. The definition of shifted twisted Yangians, based on the Drinfeld presentation, was first given in [TT24, §3.2]. Later, an equivalent parabolic presentation was given in [LPT+25, §8]. Below, we pursue the Drinfeld approach.

Given $\mu \in X_*(T)^+$, let

$$h_i(u) = u^{-2\epsilon_i^\vee(\mu)} + \hbar \sum_{r=2\epsilon_i^\vee(\mu)}^{\infty} h_i^{(r)} u^{-r-1}, \qquad b_j(u) = \hbar \sum_{r=0}^{\infty} b_j^{(r)} u^{-r-1},$$

for $i \in \tilde{\mathbb{I}}$ and $j \in \mathbb{I}$. We also set

$$\widetilde{h}_i(u) = (h_i(u))^{-1}, \qquad z_i(u) = \widetilde{h}_i(u - \tfrac{1}{2}\hbar) h_{i+1}(u).$$

Definition 5.1. *The μ-shifted twisted Yangian ${}^{\mathsf{tw}}\widetilde{Y}_{\mu,\hbar}$ of split type associated to $\mathfrak{gl}_n$ is the $\mathbb{C}[\hbar]$-algebra generated by $h_i^{(r)}, \widetilde{h}_i^{(s)}, b_j^{(t)}$ (with $i \in \tilde{\mathbb{I}}$, $j \in \mathbb{I}$, $r \geq 2\epsilon_i^\vee(\mu)$, $s \geq -2\epsilon_i^\vee(\mu)$ and $t \in \mathbb{N}$), subject to the following defining relations:*

$$z_i(u) = z_i(-u), \tag{5.1}$$

$$[h_i(u), h_j(v)] = 0, \tag{5.2}$$

$$\begin{aligned}[h_i(u), b_j(v)] = {} & \frac{\delta_{ij}\hbar}{u - v + \frac{1}{2}\hbar}\big(b_i(u + \tfrac{1}{2}\hbar) - b_i(v)\big)h_i(u) \\ & + \frac{\delta_{ij}\hbar}{u + v + \frac{1}{2}\hbar} h_i(u)\big(b_i(v) - b_i(-u - \tfrac{1}{2}\hbar)\big) \\ & + \frac{\delta_{i,j+1}\hbar}{u + v} h_i(u)\big(b_j(-u) - b_j(v)\big) \\ & + \frac{\delta_{i,j+1}\hbar}{u - v}\big(b_j(v) - b_j(u)\big)h_i(u),\end{aligned} \tag{5.3}$$

$$[b_i(u), b_j(v)] = 0 \quad \text{if } |i - j| > 1, \tag{5.4}$$

$$[b_i(u), b_i(v)] = \frac{\hbar}{v - u}\big(b_i(v) - b_i(u)\big)^2 + \frac{\hbar}{u + v}\big(\underline{z}_i(v) - \underline{z}_i(u)\big), \tag{5.5}$$

$$\begin{aligned}(u - v)[b_i(u), b_{i+1}(v)] = {} & -\tfrac{1}{2}\hbar\big(b_i(u)b_{i+1}(v) + b_{i+1}(v)b_i(u)\big) \\ & + [b_i^{(0)}, b_{i+1}(v)] + [b_{i+1}^{(0)}, b_i(u)],\end{aligned} \tag{5.6}$$

and

$$(u_1+u_2)\,\mathrm{Sym}_{u_1,u_2}\Big[b_i(u_1),\big[b_i(u_2),b_j(t)\big]\Big] = \qquad (5.7)$$
$$= 4\hbar \sum_{u=u_1,u_2} \frac{u(t-\hbar)\,\underline{z}_i(u)b_j(t) - u(t+\hbar)\,b_j(t)\underline{z}_i(u)}{4u^2-\hbar^2}$$

if $|i-j|=1$.

We denote the specialization to $\hbar=1$ as ${}^{\mathsf{tw}}\widetilde{Y}_\mu$. The usual unshifted twisted Yangian is ${}^{\mathsf{tw}}\widetilde{Y}_\hbar = {}^{\mathsf{tw}}\widetilde{Y}_{0,\hbar}$. The μ-shifted twisted Yangian ${}^{\mathsf{tw}}Y_{\mu,\hbar}$ of split type associated to $\mathfrak{sl}_n$ is the subalgebra of ${}^{\mathsf{tw}}\widetilde{Y}_{\mu,\hbar}$ generated by $b_j^{(s)}$ and $z_i^{(r)}$ ($i\in\tilde{\mathbb{I}}$, $s\geq 0$, $r\geq -2\alpha_i^\vee(\mu)$). We also write ${}^{\mathsf{tw}}Y_\mu$ for the specialization of ${}^{\mathsf{tw}}Y_{\mu,\hbar}$ at $\hbar=1$, and ${}^{\mathsf{tw}}Y_\hbar = {}^{\mathsf{tw}}Y_{0,\hbar}$.

Consider briefly the special case when $\mu=0$. Let $Z({}^{\mathsf{tw}}\widetilde{Y}_\hbar)$ denote the centre of ${}^{\mathsf{tw}}\widetilde{Y}_\hbar$.

Proposition 5.2. *There is an isomorphism*

$${}^{\mathsf{tw}}\widetilde{Y}_\hbar \cong {}^{\mathsf{tw}}Y_\hbar \otimes_{\mathbb{C}[\hbar]} Z({}^{\mathsf{tw}}\widetilde{Y}_\hbar).$$

Moreover, $Z({}^{\mathsf{tw}}\widetilde{Y}_\hbar) = \mathbb{C}[\hbar][c_1,c_3,\cdots]$*, where*

$$c(u) = 1+\hbar\sum_{r\geq 0} c_r u^{-r-1} := h_1(u)h_2(u-\hbar)\cdots h_n(u-\hbar(n-1)). \qquad (5.8)$$

Proof. See [Mol07, Theorems 2.8.2, 2.9.2]. □

5.2. **PBW theorem.** For each positive root $\beta^\vee = \alpha_j+\alpha_{j+1}+\cdots+\alpha_i$ ($j\leq i$), and $r\geq 0$, set

$$b_{\beta^\vee}^{(r)} = [b_i^{(0)},[b_{i-1}^{(0)},\cdots,[b_{j+1}^{(0)},b_j^{(r)}]\cdots]].$$

Proposition 5.3. *The algebra* ${}^{\mathsf{tw}}Y_\mu$ *has a* $\mathbb{C}$*-basis consisting of ordered monomials in the elements*

$$\{b_{\beta^\vee}^{(r)}\}_{\beta^\vee\in\Delta^+}^{r\geq 0} \cup \{z_i^{(2s_i+1)}\}_{i\in\mathbb{I}}^{s_i\geq -\alpha_i^\vee(\mu)}.$$

The same set is also a $\mathbb{C}[\hbar]$*-basis of* ${}^{\mathsf{tw}}Y_{\mu,\hbar}$*.*

Proof. This is proven in [LWZ23, Theorem 4.12] for the unshifted case, and in [LPT+25, Theorem 8.2, Proposition 8.9] and [TT24, Theorem 3.2] for the shifted case. For similar arguments in the untwisted case, see, e.g., [FKP+18, Corollary 3.15] and [FT19, Theorem 2.55]. □

As a consequence of Theorem 5.3, for any $\eta\in X_*(T)^+$ with $\eta\leq\mu$, the *shift homomorphism*

$${}^{\mathsf{tw}}\iota(\mu,\eta)\colon {}^{\mathsf{tw}}Y_{\mu,\hbar}\to {}^{\mathsf{tw}}Y_{\mu-\eta,\hbar}, \qquad b_i^{(r)}\mapsto b_i^{(r+\alpha_i^\vee(\eta))}, \quad z_i^{(r)}\mapsto z_i^{(r+2\alpha_i^\vee(\eta))} \qquad (5.9)$$

is injective. If $\eta=\mu$, we abbreviate ${}^{\mathsf{tw}}\iota(\mu) = {}^{\mathsf{tw}}\iota(\mu,\mu)$. In particular, ${}^{\mathsf{tw}}Y_{\mu,\hbar}$ can be realized as a subalgebra of the unshifted twisted Yangian ${}^{\mathsf{tw}}Y_\hbar$ via ${}^{\mathsf{tw}}\iota(\mu)$.

5.3. **Canonical filtration.** We are primarily interested in the following subalgebra of the shifted twisted Yangian.

Definition 5.4. *Let* ${}^{\mathsf{tw}}\mathbf{Y}_\mu$ *be the* $\mathbb{C}[\hbar]$*-subalgebra of* ${}^{\mathsf{tw}}Y_{\mu,\hbar}$ *generated by*

$$\{\hbar b_{\beta^\vee}^{(r)}\}_{\beta^\vee\in\Delta^+}^{r\geq 0} \cup \{\hbar z_i^{(2s_i+1)}\}_{i\in\mathbb{I}}^{s_i\geq -\alpha_i^\vee(\mu)}.$$

We will now give an alternative description of ${}^{\mathsf{tw}}\mathbf{Y}_\mu$ as a Rees algebra, in analogy to the untwisted case. There is a filtration $F_\mu^\bullet$ on ${}^{\mathsf{tw}}Y_{\mu,\hbar}$ determined by

$$\deg b_{\beta^\vee}^{(r)} = \beta^\vee(\mu) + r + 1, \qquad \deg z_i^{(r)} = 2\alpha_i^\vee(\mu) + r + 1.$$

Theorem 5.5. *For any* $\mu \in X_*(T)^+$*, there is a canonical* $\mathbb{C}[\hbar]$*-algebra isomorphism*

$${}^{\mathsf{tw}}\mathbf{Y}_\mu \cong \mathrm{Rees}^{F_\mu^\bullet}\, {}^{\mathsf{tw}}Y_\mu.$$

Proof. The proof relies on a comparison of Rees algebras with respect to the loop and canonical filtrations, and is formally the same as the proof of [FT19, Theorem A.32]. □

Passing to the semi-classical limit, there are canonical isomorphisms

$${}^{\mathsf{tw}}\mathbf{Y}_\mu/\hbar{}^{\mathsf{tw}}\mathbf{Y}_\mu \cong \mathrm{Rees}^{F_\mu^\bullet}\, {}^{\mathsf{tw}}Y_\mu/\hbar\, \mathrm{Rees}^{F_\mu^\bullet}\, {}^{\mathsf{tw}}Y_\mu \cong \mathrm{gr}^{F_\mu^\bullet}\, {}^{\mathsf{tw}}Y_\mu.$$

Moreover, by [TT24, Lemma 3.3], $\mathrm{gr}^{F_\mu^\bullet}\, {}^{\mathsf{tw}}Y_\mu$ is a commutative algebra. It is also naturally a Poisson algebra, with the Poisson bracket given by

$$\{a, b\} = \hbar^{-1}[\hat{a}, \hat{b}] \mod \hbar{}^{\mathsf{tw}}\mathbf{Y}_\mu,$$

for any lifts $\hat{a}, \hat{b}$. Finally, we remark that the shift homomorphisms (5.9) induce monomorphisms of Poisson algebras

$${}^{\mathsf{tw}}\iota(\mu) : {}^{\mathsf{tw}}\mathbf{Y}_\mu/\hbar{}^{\mathsf{tw}}\mathbf{Y}_\mu \hookrightarrow {}^{\mathsf{tw}}\mathbf{Y}_{\mu-\eta}/\hbar{}^{\mathsf{tw}}\mathbf{Y}_{\mu-\eta}.$$

5.4. **Reflection equation presentation.** The *RTT twisted Yangian* ${}^{\mathsf{tw}}\widetilde{Y}_\hbar^{\mathsf{rtt}}$ of split type associated to $\mathfrak{gl}_n$ is the algebra generated by $s_{ij}^{(k)}$, where $k \geq 1$ and $1 \leq i, j \leq n$, subject to the relations

$$\begin{aligned}(u^2 - v^2)[s_{ij}(u), s_{kl}(v)] &= \hbar(u+v)(s_{kj}(u)s_{il}(v) - s_{kj}(v)s_{il}(u)) \\ &\quad - \hbar(u-v)(s_{ik}(u)s_{jl}(v) - s_{ki}(v)s_{lj}(u)) \\ &\quad + \hbar^2(s_{ki}(u)s_{jl}(v) - s_{ki}(v)s_{jl}(u)) \\ s_{ji}(-u) &= s_{ij}(u) + \hbar\frac{s_{ij}(u) - s_{ij}(-u)}{2u}.\end{aligned}$$

Writing

$$S(u) = \sum_{i,j=1}^n e_{ij} \otimes s_{ij}(u), \qquad s_{ij}(u) = \delta_{ij} + \sum_{k>0} s_{ij}^{(k)} u^{-k}$$

these relations can also be presented in matrix form:

$$R(u-v)S_1(u)R^t(-u-v)S_2(v) = S_2(v)R^t(-u-v)S_1(u)R(u-v),$$
$$S^t(-u) = S(u) + \hbar\frac{S(u)-S(-u)}{2u}.$$

By [Mol07, Theorem 2.4.3], there is an injective algebra homomorphism

$$^{\mathsf{tw}}\widetilde{Y}_\hbar^{\mathsf{rtt}} \hookrightarrow \widetilde{Y}_\hbar^{\mathsf{rtt}}, \qquad S(u) \mapsto T^t(-u)T(u).$$

Moreover, by [Mol07, Theorem 2.10.1], this embedding endows $^{\mathsf{tw}}\widetilde{Y}_\hbar^{\mathsf{rtt}}$ with the structure of a right coideal subalgebra.

Theorem 5.6 ([Mol07, Corollary 2.4.4, Remark 2.4.5]). *Ordered monomials in the PBW variables*

$$\{s_{ij}^{(r)}\}_{i>j}^{r\geq 1} \cup \{s_{ii}^{(2r)}\}_{i\in\mathbb{I}}^{r\geq 1}$$

form a basis of $^{\mathsf{tw}}\widetilde{Y}_\hbar^{\mathsf{rtt}}$ *as a free* $\mathbb{C}[\hbar]$*-module, as well as a basis of* $^{\mathsf{tw}}\widetilde{Y}^{\mathsf{rtt}}$ *as a free* $\mathbb{C}$*-module.*

We rely on the following key result linking the RTT and current realizations of the twisted Yangian.

Theorem 5.7 ([LWZ23, Theorem 5.1]). *There is a canonical isomorphism*

$$\Upsilon\colon {}^{\mathsf{tw}}\widetilde{Y}_\hbar \xrightarrow{\sim} {}^{\mathsf{tw}}\widetilde{Y}_\hbar^{\mathsf{rtt}} \tag{5.10}$$

from the twisted Yangian $^{\mathsf{tw}}\widetilde{Y}_\hbar$ *in the Drinfeld presentation (Definition 5.1) to the RTT twisted Yangian* $^{\mathsf{tw}}\widetilde{Y}_\hbar^{\mathsf{rtt}}$*, given by Gauss decomposition.*

Let $^{\mathsf{tw}}\widetilde{\mathbf{Y}}^{\mathsf{rtt}}$ be the $\mathbb{C}[\hbar]$-subalgebra of $^{\mathsf{tw}}\widetilde{Y}_\hbar^{\mathsf{rtt}}$ generated by $\{\hbar s_{ij}^{(r)}\}_{i,j\in\tilde{\mathbb{I}}}^{r\geq 1}$. Recall that, when $\mu = 0$, we abbreviate $^{\mathsf{tw}}\widetilde{\mathbf{Y}} = {}^{\mathsf{tw}}\widetilde{\mathbf{Y}}_0$.

Proposition 5.8. *The isomorphism (5.10) restricts to an isomorphism*

$$\Upsilon\colon {}^{\mathsf{tw}}\widetilde{\mathbf{Y}} \xrightarrow{\sim} {}^{\mathsf{tw}}\widetilde{\mathbf{Y}}^{\mathsf{rtt}}.$$

Proof. The proof is entirely analogous to the proof of [FT19, Proposition 2.34]. □

5.5. **Ciccoli-Gavarini duality.** When $\mu = 0$, the algebra $^{\mathsf{tw}}\mathbf{Y}_0 = {}^{\mathsf{tw}}\mathbf{Y}_0$ can be identified with the Ciccoli–Gavarini dual of $^{\mathsf{tw}}Y_\hbar$. Let us first recall the general definition of this concept. We continue to use the notation from §4.4.

Let $\mathfrak{b} \subset \mathfrak{a}$ be a Lie coideal, and suppose that B is a *compatible* deformation–quantization of $U(\mathfrak{b})$, i.e., $B \subset A$ is a coideal subalgebra over $\mathbb{C}[\hbar]$, and there is an isomorphism of Hopf algebras $B/\hbar B \cong U(\mathfrak{b})$. The *Ciccoli–Gavarini dual* $B^{\daleth}$ of B is the following coideal subalgebra of A':

$$B^{\daleth} = \{a \in B \mid \delta_m(a) \in \hbar^m A^{\otimes(m-1)} \otimes B \text{ for all } m \in \mathbb{N}\} = B \cap A'.$$

The importance of $B^\daleth$ stems from the fact that, according to the quantum duality principle [CG06, Theorem 3.3], $B^\daleth$ is a deformation–quantization of the coordinate ring of the homogeneous space $G_{\mathfrak{a}^*}/G_{\mathfrak{b}^\perp}$.

Let us now return to the specific setting of the twisted Yangian ${}^{\mathsf{tw}}Y_\hbar$. It is a coideal subalgebra of $Y_\hbar$, and admits a grading with $\deg(\hbar) = 1$ and $\deg(x^{(r)}) = r$, for $x = z_i, b_{\beta^\vee}$ (or $\deg s_{ij}^{(r)} = r - 1$ in the RTT presentation). The description of the coideal algebra structure can be found in, e.g., [Mol07, (2.64)]. The twisted Yangian is a deformation–quantization of the universal enveloping algebra of the twisted current Lie algebra, i.e., ${}^{\mathsf{tw}}Y_\hbar/\hbar{}^{\mathsf{tw}}Y_\hbar \cong U(\mathfrak{sl}_n[t]^\sigma)$, where σ is the composition of the Cartan involution with the map $t \mapsto -t$. This isomorphism is an isomorphism of graded Hopf algebras if $U(\mathfrak{sl}_n[t]^\sigma)$ is endowed with the loop grading.

Theorem 5.9. *There is a canonical* $\mathbb{C}[\hbar]$*-algebra isomorphism*

$$ {}^{\mathsf{tw}}Y_\hbar^\daleth \cong {}^{\mathsf{tw}}\mathbf{Y}_0. $$

Proof. The claim will follow from Proposition 5.8 if we can show ${}^{\mathsf{tw}}\widetilde{Y}_\hbar^\daleth \cong {}^{\mathsf{tw}}\widetilde{\mathbf{Y}}^{\mathsf{rtt}}$. For the latter, one can apply the same argument as in [FT19, Theorem A.26]. □

6. Sklyanin minors and ABCD presentation

Drinfeld's new realization of the Yangian admits a natural description in terms of quantum minors [Dri87a, NT94, GKLO05]. Below we deduce an analogous description of the current realization of the twisted Yangian from [LWZ23] in terms of *Sklyanin minors*. We assume $\mu = 0$ throughout.

6.1. **Current generators as Sklyanin minors.** For the sake of conciseness, we freely use the standard definition, properties and notation for the Sklyanin determinant and Sklyanin minors, as in, e.g., [Mol07, §2.5–§2.6], without recalling these in detail.

Recall the series $f_i(u), e_i(u), d_i(u)$ constructed by Gauss decomposition in [LWZ23, (3.1)–(3.3)]. The following proposition allows us to express the current generators of the twisted Yangian in terms of Sklyanin minors.

Proposition 6.1. *We have:*

$$
\begin{aligned}
f_i(u) &= s^{1\dots i-1,i+1}_{1\dots i}(u + \hbar(i-1))(s^{1\dots i}_{1\dots i}(u + \hbar(i-1)))^{-1}, \\
e_i(u) &= (s^{1\dots i}_{1\dots i}(u + \hbar(i-1)))^{-1} s^{1\dots i}_{1\dots i-1,i+1}(u + \hbar(i-1)), \\
d_i(u) &= s^{1\dots i}_{1\dots i}(u + \hbar(i-1))(s^{1\dots i-1}_{1\dots i-1}(u + \hbar(i-1)))^{-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
b_i(u) &= f_i(u - \tfrac{\hbar i}{2}) = s^{1\dots i-1,i+1}_{1\dots i}(u + \tfrac{\hbar i}{2} - \hbar)(s^{1\dots i}_{1\dots i}(u + \tfrac{\hbar i}{2} - \hbar))^{-1}, \\
z_i(u) &= \frac{d_{i+1}(u - \frac{\hbar i}{2})}{d_i(u - \frac{\hbar i}{2})} = \frac{s^{1\dots i-1}_{1\dots i-1}(u + \frac{\hbar i}{2} - \hbar) s^{1\dots i+1}_{1\dots i+1}(u + \frac{\hbar i}{2})}{s^{1\dots i}_{1\dots i}(u + \frac{\hbar i}{2} - \hbar) s^{1\dots i}_{1\dots i}(u + \frac{\hbar i}{2})}.
\end{aligned}
$$

Proof. This follows immediately from the formulae in the proof of [LWZ23, Corollary 3.2] and [Mol07, Lemma 2.14.2]. □

We remark that, since $e_i(u) = f_i(-u - \hbar i)$, we also get $b_i(-u) = e_i(u - \frac{\hbar i}{2})$.

6.2. **Relations.** In this subsection, we will describe some of the relations between the Sklyanin minors. Define:

$$A_i(u) = s^{1\dots i}_{1\dots i}(u + \tfrac{\hbar i}{2} - \hbar), \qquad B_i(u) = s^{1\dots i}_{1\dots i-1,i+1}(u + \tfrac{\hbar i}{2} - \hbar),$$
$$C_i(u) = s^{1\dots i-1,i+1}_{1\dots i}(u + \tfrac{\hbar i}{2} - \hbar), \qquad D_i(u) = s^{1\dots i-1,i+1}_{1\dots i-1,i+1}(u + \tfrac{\hbar i}{2} - \hbar).$$

Then, by Proposition 6.1,

$$B_i(u) = A_i(u)b_i(-u), \quad C_i(u) = b_i(u)A_i(u), \quad z_i(u) = \frac{A_{i+1}(u + \frac{1}{2}\hbar)A_{i-1}(u + \frac{1}{2}\hbar)}{A_i(u)A_i(u + \hbar)}.$$

Note that $A_i(u)$ satisfies $A_i(-u) = A_i(u + \hbar)$. If we set $\widetilde{A}_i(u) = A_i(u + \frac{1}{2}\hbar)$, then $\widetilde{A}_i(-u) = \widetilde{A}_i(u)$ and

$$z_i(u) = \frac{\widetilde{A}_{i+1}(u)\widetilde{A}_{i-1}(u)}{\widetilde{A}_i(u - \frac{1}{2}\hbar)\widetilde{A}_i(u + \frac{1}{2}\hbar)}. \tag{6.1}$$

We will similarly denote $\widetilde{X}_i(u) = X_i(u + \frac{1}{2}\hbar)$ for $X \in \{B, C, D\}$.

Recall that the Sklyanin comatrix $\widehat{S}(u)$ is defined by

$$\widehat{S}(u)S(u - n + \hbar) = \operatorname{sdet} S(u).$$

Lemma 6.2. *The following entries of the Sklyanin comatrix are equal to the corresponding Sklyanin minors:*

$$\widehat{s}_{n-1,n-1}(u) = s^{1\dots n-2,n}_{1\dots n-2,n}(u), \qquad \widehat{s}_{n,n}(u) = s^{1\dots n-1}_{1\dots n-1}(u),$$
$$\widehat{s}_{n-1,n}(u) = s^{1\dots n-2,n}_{1\dots n-1}(u), \qquad \widehat{s}_{n,n-1}(u) = s^{1\dots n-1}_{1\dots n-2,n}(u).$$

Proof. This is proven in the same way as [JZ24, Proposition 4.1]. □

Lemma 6.3 ([Mol07, Proposition 2.12.3])**.** *The map*

$$S(u) \mapsto \widehat{S}(-u + \tfrac{\hbar n}{2} - \hbar)$$

is an automorphism.

Let $\dot{X}_i(u) = X_i(-u + \frac{1}{2}\hbar)$ for $X \in \{A, B, C, D\}$. Applying the two lemmas above, we get the following analogue of [GKLO05, Proposition 2.1] and [NT94, Proposition 1.2]. Note that the list below is not necessarily exhaustive, i.e., we expect more relations are needed to get a full presentation.

Proposition 6.4. *The following relations hold:*

$$\begin{aligned}
(u^2-v^2)[\dot{A}_i(u),\dot{B}_i(v)] &= \hbar(u+v)(\dot{A}_i(u)\dot{B}_i(v)-\dot{A}_i(v)\dot{B}_i(u))\\
&\quad -\hbar(u-v)(\dot{A}_i(u)\dot{B}_i(v)-\dot{A}_i(v)\dot{C}_i(u))\\
&\quad +\hbar^2(\dot{A}_i(u)\dot{B}_i(v)-\dot{A}_i(v)\dot{B}_i(u)),\\
(u^2-v^2)[\dot{A}_i(u),\dot{C}_i(v)] &= \hbar(u+v)(\dot{A}_i(u)\dot{C}_i(v)-\dot{A}_i(v)\dot{C}_i(u))\\
&\quad -\hbar(u-v)(\dot{A}_i(u)\dot{C}_i(v)-\dot{A}_i(v)\dot{B}_i(u))\\
&\quad +\hbar^2(\dot{A}_i(u)\dot{C}_i(v)-\dot{A}_i(v)\dot{C}_i(u)),\\
(u^2-v^2)[\dot{B}_i(u),\dot{C}_i(v)] &= \hbar(u+v)(\dot{A}_i(u)\dot{D}_i(v)-\dot{A}_i(v)\dot{D}_i(u))\\
&\quad -\hbar(u-v)(\dot{B}_i(u)\dot{C}_i(v)-\dot{C}_i(v)\dot{B}_i(u))\\
&\quad -\hbar^2(u+v)^{-1}(\dot{A}_i(u)\dot{D}_i(v)-\dot{A}_i(v)\dot{D}_i(u)),\\
\dot{A}_i(-u) &= \dot{A}_i(u),\\
\dot{B}_i(-u) &= \dot{C}_i(u)+\hbar\frac{\dot{C}_i(u)-\dot{C}_i(-u)}{2u},\\
\dot{C}_i(-u) &= \dot{B}_i(u)+\hbar\frac{\dot{B}_i(u)-\dot{B}_i(-u)}{2u},\\
(u+v)[\dot{C}_i(u),\dot{C}_i(v)] &= \hbar\big(\dot{A}_i(v)\dot{D}_i(u)-\dot{A}_i(u)\dot{D}_i(v)\big),\\
(u+v)[\dot{B}_i(u),\dot{B}_i(v)] &= \hbar\big(\dot{D}_i(v)\dot{A}_i(u)-\dot{D}_i(u)\dot{A}_i(v)\big),\\
[\dot{A}_i(u),\dot{A}_i(v)] &= 0.
\end{aligned}$$

Proof. This follows from Lemmas 6.2–6.3 and the RTT relations in the twisted Yangian (see, e.g., [Mol07, Proposition 2.2.1]). ☐

7. Twisted GKLO representations

In [GKLO05], a remarkable family of representations of Yangians, quantizing moduli spaces of monopoles, was introduced by Gerasimov, Kharchev, Lebedev and Oblezin. Today they are commonly known as 'GKLO representations'. They were later generalized to the cases of dominantly [KWWY14] and arbitrarily [BFN19] shifted Yangians. Below we construct analogous representations for the shifted *twisted* Yangians ${}^{\mathsf{tw}}\widetilde{Y}_{\mu,\hbar}$.

7.1. **Difference operators.** Fix a dominant coweight $\lambda \in X_*(T)^+$ with $\mu \leq \lambda$, and set $m_i = \omega_i^\vee(\lambda-\mu)$ and $\lambda_i = \alpha_i^\vee(\lambda)$. In other words,

$$\lambda = \sum_{i=1}^{n-1}\lambda_i\omega_i, \qquad \mu = \sum_{i=1}^{n-1}\mu_i\omega_i, \qquad \lambda-\mu = \sum_{i=1}^{n-1} m_i\alpha_i.$$

Note that, since $\mu \leq \lambda$, each m_i is a non-negative integer. Moreover, we have the identity

$$\lambda_i - \mu_i = 2m_i - \sum_{j\sim i} m_j, \tag{7.1}$$

where $j \sim i$ means that j and i are connected by an edge in the Dynkin diagram. Let $D^\lambda_{\mu,\hbar}$ be the $\mathbb{C}[\hbar]$-algebra generated by

- $\gamma_{i,k}, \beta^{\pm 1}_{i,k}$ (for $i \in \mathbb{I}$ and $1 \leq k \leq m_i$),
- $((\gamma_{i,k} + r\hbar)^2 - (\gamma_{i,l} + s\hbar)^2)^{-1}$, (for $k \neq l$ and $r, s \in \mathbb{Z}$),
- $(\gamma_{i,k} \pm \frac{r}{2}\hbar)^{-1}$ (for $r \in \mathbb{Z}$),

subject to the relations

$$[\beta^{\pm 1}_{i,k}, \gamma_{j,l}] = \pm\delta_{ij}\delta_{kl}\hbar\beta^{\pm 1}_{i,k}, \qquad [\gamma_{i,k}, \gamma_{j,l}] = 0 = [\beta_{i,k}, \beta_{j,l}], \qquad \beta^{\pm 1}_{i,k}\beta^{\mp 1}_{i,k} = 1.$$

The algebra $D^\lambda_{\mu,\hbar}$ has a natural representation on the space

$$\mathrm{Pol}^\lambda_{\mu,\hbar} = \mathbb{C}[\hbar][\gamma_{i,k}, ((\gamma_{i,k} + r\hbar)^2 - (\gamma_{i,l} + s\hbar)^2)^{-1}, (\gamma_{i,k} + \tfrac{r}{2}\hbar)^{-1}]^{r,s\in\mathbb{Z}}_{i\in\mathbb{I},1\leq k\neq l\leq m_i},$$

where $\gamma_{i,k}$ acts by multiplication and $\beta^{\pm 1}_{i,k}$ acts by the difference operator $e^{\pm\hbar\partial_{\gamma_{i,k}}}$. In analogy to [KWWY14, Proposition 4.4], there is an isomorphism of Poisson algebras

$$D^\lambda_{\mu,\hbar}/\hbar D^\lambda_{\mu,\hbar} \cong \mathbb{C}[\gamma_{i,k}, \beta^{\pm 1}_{i,k}, ((\gamma_{i,k} + r\hbar)^2 - (\gamma_{i,l} + s\hbar)^2)^{-1}, (\gamma_{i,k} \pm \tfrac{r}{2}\hbar)^{-1}]^{r,s\in\mathbb{Z}}_{i\in\mathbb{I},1\leq k\neq l\leq m_i},$$

where the only non-trivial Poisson bracket between the generators on the RHS is $\{\beta^{\pm 1}_{i,k}, \gamma_{i,k}\} = \pm\beta^{\pm 1}_{i,k}$.

7.2. **Auxiliary relations.** We begin by defining some auxiliary operators in $D^\lambda_{\mu,\hbar}$ and describing the relations between them. Let us abbreviate $\xi_{i,k} = \gamma_{i,k} + \hbar$. Choose polynomials

$$R_i(u) = \prod_{k=1}^{\lambda_i}(u^2 - r^2_{i,k}) \qquad (i \in \mathbb{I}),$$

where $r_{i,k}$ are arbitrary complex numbers. Define

$$\varkappa_{i,k} = \frac{\prod_{l=1}^{m_{i+1}}(\gamma^2_{i,k} - (\gamma_{i+1,l} + \frac{1}{2}\hbar)^2)\prod_{l=1}^{m_{i-1}}(\gamma_{i,k} + \gamma_{i-1,l} + \frac{1}{2}\hbar)}{2(\gamma_{i,k} - \frac{1}{2}\hbar)\prod_{l\neq k}(\gamma^2_{i,k} - \gamma^2_{i,l})}\beta^{-1}_{i,k} \in D^\lambda_{\mu,\hbar}, \tag{7.2}$$

$$\varkappa'_{i,k} = R_i(\xi_{i,k})\frac{\prod_{l=1}^{m_{i-1}}(\xi_{i,k} - (\gamma_{i-1,l} + \frac{1}{2}\hbar))}{2(\gamma_{i,k} + \frac{3}{2}\hbar)\prod_{l\neq k}(\xi^2_{i,k} - \xi^2_{i,l})}\beta_{i,k} \in D^\lambda_{\mu,\hbar}. \tag{7.3}$$

The following lemma follows easily by direct calculation.

Lemma 7.1. *The following relations hold:*

$$[\varkappa_{i,k},\gamma_{j,l}] = -\hbar\delta_{ij}\delta_{kl}\varkappa_{i,k}, \qquad [\varkappa'_{i,k},\gamma_{j,l}] = \hbar\delta_{ij}\delta_{kl}\varkappa'_{i,k}, \tag{7.4}$$

$$[\varkappa_{i,k},\varkappa_{j,l}] = \frac{\frac{1}{2}a_{ij}\hbar}{\gamma_{i,k}-\gamma_{j,l}}[\varkappa_{j,l},\varkappa_{i,k}]_+, \qquad [\varkappa'_{i,k},\varkappa'_{j,l}] = [\varkappa'_{j,l},\varkappa'_{i,k}]_+\frac{-\frac{1}{2}a_{ij}\hbar}{\gamma_{i,k}-\gamma_{j,l}}, \tag{7.5}$$

$$[\varkappa_{i,k},\varkappa'_{j,l}] = \frac{\frac{1}{2}a_{ij}\hbar}{\gamma_{i,k}+\gamma_{j,l}+\hbar}[\varkappa_{i,k},\varkappa'_{j,l}]_+ \qquad \big((i,k)\neq(j,l)\big), \tag{7.6}$$

$$\varkappa_{i,k}\varkappa'_{i,k} = R_i(\gamma_{i,k})\frac{\prod_{j=i\pm1}\prod_{l=1}^{m_j}(\gamma_{i,k}^2-(\gamma_{j,l}+\frac{1}{2}\hbar)^2)}{4(\gamma_{i,k}-\frac{1}{2}\hbar)(\gamma_{i,k}+\frac{1}{2}\hbar)\prod_{l\neq k}(\gamma_{i,k}^2-\gamma_{i,l}^2)(\gamma_{i,k}^2-\xi_{i,l}^2)}, \tag{7.7}$$

$$\varkappa'_{i,k}\varkappa_{i,k} = R_i(\xi_{i,k})\frac{\prod_{j=i\pm1}\prod_{l=1}^{m_j}(\xi_{i,k}^2-(\gamma_{j,l}+\frac{1}{2}\hbar)^2)}{4(\xi_{i,k}-\frac{1}{2}\hbar)(\xi_{i,k}+\frac{1}{2}\hbar)\prod_{l\neq k}(\xi_{i,k}^2-\gamma_{i,l}^2)(\xi_{i,k}^2-\xi_{i,l}^2)}. \tag{7.8}$$

Relations (7.5)–(7.6) can also be reformulated as

$$(\gamma_{i,k}-\gamma_{j,l}-\tfrac{1}{2}a_{ij}\hbar)\varkappa_{i,k}\varkappa_{j,l} = (\gamma_{i,k}-\gamma_{j,l}+\tfrac{1}{2}a_{ij}\hbar)\varkappa_{j,l}\varkappa_{i,k}, \tag{7.9}$$

$$\varkappa'_{i,k}\varkappa'_{j,l}(\gamma_{i,k}-\gamma_{j,l}+\tfrac{1}{2}a_{ij}\hbar) = \varkappa'_{j,l}\varkappa'_{i,k}(\gamma_{i,k}-\gamma_{j,l}-\tfrac{1}{2}a_{ij}\hbar), \tag{7.10}$$

$$(\gamma_{i,k}+\gamma_{j,l}+(1-\tfrac{1}{2}a_{ij})\hbar)\varkappa_{i,k}\varkappa'_{j,l} = (\gamma_{i,k}+\gamma_{j,l}+(1+\tfrac{1}{2}a_{ij})\hbar)\varkappa'_{j,l}\varkappa_{i,k}. \tag{7.11}$$

For later convenience, for $i\neq j$, abbreviate

$$S_{i,k}^{j,l} = \frac{4(\gamma_{i,k}-\frac{1}{2}\hbar)(\gamma_{i,k}+\frac{1}{2}\hbar)\varkappa_{i,k}\varkappa'_{i,k}}{\gamma_{i,k}^2-(\gamma_{j,l}+\frac{1}{2}\hbar)^2}, \qquad T_{i,k}^{j,l} = \frac{4(\xi_{i,k}-\frac{1}{2}\hbar)(\xi_{i,k}+\frac{1}{2}\hbar)\varkappa'_{i,k}\varkappa_{i,k}}{\xi_{i,k}^2-(\xi_{j,l}+\frac{1}{2}\hbar)^2}.$$

7.3. **Twisted GKLO homomorphism.** Below we define an analogue of the *GKLO homomorphism* for shifted twisted Yangians.

Theorem 7.2. *The assignment*

$$b_i(u) \mapsto \sum_{k=1}^{m_i}\frac{1}{u-\gamma_{i,k}}\varkappa_{i,k} + \frac{1}{u+\gamma_{i,k}+\hbar}\varkappa'_{i,k}, \tag{7.12}$$

$$h_i(u) \mapsto u^{-4m_1}\prod_{j=1}^{i-1}R_j(u-\tfrac{i-1-j}{2}\hbar)\frac{\prod_{k=1}^{m_i}(u^2-(\gamma_{i,k}+\frac{1}{2}\hbar)^2)}{\prod_{l=1}^{m_{i-1}}(u-\gamma_{i-1,l})(u+\gamma_{i-1,l}+\hbar)} \tag{7.13}$$

defines a homomorphism $\Phi_\mu^\lambda\colon {}^{\mathsf{tw}}\widetilde{Y}_{\mu,\hbar}\to D_{\mu,\hbar}^\lambda$.

We define the *λ-truncated μ-shifted twisted Yangian* ${}^{\mathsf{tw}}\widetilde{Y}_{\mu,\hbar}^\lambda$ to be the image of ${}^{\mathsf{tw}}\widetilde{Y}_{\mu,\hbar}$ under Φ_μ^λ. We refer to the pullback by Φ_μ^λ of the natural representation $\mathrm{Pol}_{\mu,\hbar}^\lambda$ of $D_{\mu,\hbar}^\lambda$ as a *twisted GKLO representation.* Restriction also yields representations of ${}^{\mathsf{tw}}Y_{\mu,\hbar}$ and ${}^{\mathsf{tw}}\mathbf{Y}_\mu$. Let ${}^{\mathsf{tw}}Y_{\mu,\hbar}^\lambda$ (resp. ${}^{\mathsf{tw}}\mathbf{Y}_\mu^\lambda$) denote the image of ${}^{\mathsf{tw}}Y_{\mu,\hbar}$ (resp. ${}^{\mathsf{tw}}\mathbf{Y}_\mu$) under Φ_μ^λ.

7.4. **Relations check.** We prove Theorem 7.2 by explicitly checking that Φ^λ_μ preserves the relations from Definition 5.1. Relations (5.2) and (5.4) are immediate.

7.4.1. *Range of powers.* We first check that the operators on the RHS of (7.12)–(7.13), when expanded in u^{-1}, have the correct range of powers. The RHS of (7.12) lies in $u^{-1}D^\lambda_{\mu,\hbar}[[u^{-1}]]$, matching $b_i(u)$. On the other hand, the RHS of (7.13) lies in $u^p D^\lambda_{\mu,\hbar}[[u^{-1}]]$, where

$$p = 2\left(\sum_{j=1}^{i-1}\lambda_j + m_i - m_{i-1}\right) - 4m_1 = -2m_1 + 2\sum_{j=1}^{i}\mu_j = -2\epsilon_i^\vee(\mu),$$

by (7.1), and the coefficient at the top power is 1, which matches $h_i(u)$.

7.4.2. *Relation* (5.1). It follows directly from (7.13) that

$$z_i(u) = (h_i(u - \tfrac{1}{2}\hbar))^{-1}h_{i+1}(u) \mapsto R_i(u)\frac{\prod_{j=i\pm1}\prod_{k=1}^{m_j}(u^2 - (\gamma_{j,k} + \frac{1}{2}\hbar)^2)}{\prod_{k=1}^{m_i}(u^2 - \gamma_{i,k}^2)(u^2 - \xi_{i,k}^2)}. \tag{7.14}$$

Since (7.14) involves only even powers of u, (5.1) follows.

7.4.3. *Relation* (5.3). Given (7.4), it suffices to prove (5.3) in the case where $m_j = 1$, $\lambda_j = 0$ and $m_{j\pm1} = 0$. Therefore, in this subsection we will omit the second subscript on γ and $\varkappa$.

First consider the case $i = j$. Let us calculate the LHS of (5.3):

$$h_i(u)b_i(v) = \frac{(u^2 - (\gamma_i + \frac{1}{2}\hbar)^2)}{(v - \gamma_i)}\varkappa_i + \frac{(u^2 - (\gamma_i + \frac{1}{2}\hbar)^2)}{(v + \gamma_i + \hbar)}\varkappa_i',$$

$$b_i(v)h_i(u) = \frac{(u^2 - (\gamma_i - \frac{1}{2}\hbar)^2)}{(v - \gamma_i)}\varkappa_i + \frac{(u^2 - (\gamma_i + \frac{3}{2}\hbar)^2)}{(v + \gamma_i + \hbar)}\varkappa_i',$$

$$[h_i(u), b_i(v)] = \frac{-2\hbar\gamma_i}{(v - \gamma_i)}\varkappa_i + \frac{2\hbar(\gamma_i + \hbar)}{(v + \gamma_i + \hbar)}\varkappa_i'.$$

Next, we compute the RHS:

$$h_i(u)(b_i(v) - b_i(-u - \tfrac{1}{2}\hbar)) = \frac{(u^2 - (\gamma_i + \frac{1}{2}\hbar)^2)(u + v + \frac{1}{2}\hbar)}{(v - \gamma_i)(u + \gamma_i + \frac{1}{2}\hbar)}\varkappa_i + \frac{(u^2 - (\gamma_i + \frac{1}{2}\hbar)^2)(u + v + \frac{1}{2}\hbar)}{(v + \gamma_i + \hbar)(u - \gamma_i - \frac{1}{2}\hbar)}\varkappa_i',$$

$$(b_i(v) - b_i(u + \tfrac{1}{2}\hbar))h_i(u) = \frac{(u^2 - (\gamma_i - \frac{1}{2}\hbar)^2)(u - v + \frac{1}{2}\hbar)}{(v - \gamma_i)(u - \gamma_i + \frac{1}{2}\hbar)}\varkappa_i + \frac{(u^2 - (\gamma_i + \frac{3}{2}\hbar)^2)(u - v + \frac{1}{2}\hbar)}{(v + \gamma_i + \hbar)(u + \gamma_i + \frac{3}{2}\hbar)}\varkappa_i'.$$

Hence

$$\frac{\hbar h_i(u)(b_i(v)-b_i(-u-\frac{1}{2}\hbar))}{u+v+\frac{1}{2}\hbar}-\frac{\hbar(b_i(v)-b_i(u+\frac{1}{2}\hbar))h_i(u)}{u-v+\frac{1}{2}\hbar}=\frac{-2\hbar\gamma_i}{(v-\gamma_i)}\varkappa_i+\frac{2\hbar(\gamma_i+\hbar)}{(v+\gamma_i+\hbar)}\varkappa_i',$$

as required.

Now consider the case $i=j+1$. Let us calculate the LHS of (5.3):

$$\begin{aligned}
h_{j+1}(u)b_j(v) &= \frac{1}{(v-\gamma_j)(u-\gamma_j)(u+\gamma_j+\hbar)}\varkappa_j+\frac{1}{(v+\gamma_j+\hbar)(u-\gamma_j)(u+\gamma_j+\hbar)}\varkappa_j',\\
b_j(v)h_{j+1}(u) &= \frac{1}{(v-\gamma_j)(u-\gamma_j+\hbar)(u+\gamma_j)}\varkappa_j+\frac{1}{(v+\gamma_j+\hbar)(u-\gamma_j-\hbar)(u+\gamma_j+2\hbar)}\varkappa_j',\\
[h_{j+1}(u),b_j(v)] &= \frac{2\hbar\gamma_j}{(v-\gamma_j)(u^2-\gamma_j^2)((u+\hbar)^2-\gamma_j^2)}\varkappa_j\\
&\quad -\frac{2\hbar(\gamma_j+\hbar)}{(v+\gamma_j-\hbar)(u^2-(\gamma_j+\hbar)^2)(u-\gamma_j)(u+\gamma_j+2\hbar)}\varkappa_j'.
\end{aligned}$$

Next, we compute the RHS:

$$\begin{aligned}
h_j(u)\big(b_j(v)-b_j(-u)\big) &= \frac{u+v}{(v-\gamma_j)(u^2-\gamma_j^2)(u+\gamma_j+\hbar)}\varkappa_j\\
&\quad+\frac{u+v}{(v+\gamma_j+\hbar)(u-\gamma_j)(u^2-(\gamma_j+\hbar)^2)}\varkappa_j',\\
\big(b_j(v)-b_j(u)\big)h_j(u) &= \frac{u-v}{(v-\gamma_j)(u^2-\gamma_j^2)(u-\gamma_j+\hbar)}\varkappa_j\\
&\quad+\frac{u-v}{(v+\gamma_j+\hbar)(u^2-(\gamma_j+\hbar)^2)(u+\gamma_j+2\hbar)}\varkappa_j'.
\end{aligned}$$

Hence

$$\frac{\hbar(b_j(v)-b_j(u))h_j(u)}{u-v}-\frac{\hbar h_j(u)(b_j(v)-b_j(-u))}{u+v}=$$

$$=\frac{2\hbar\gamma_j}{(v-\gamma_j)(u^2-\gamma_j^2)((u+\hbar)^2-\gamma_j^2)}\varkappa_j-\frac{2\hbar(\gamma_j+\hbar)}{(v+\gamma_j-\hbar)(u^2-(\gamma_j+\hbar)^2)(u-\gamma_j)(u+\gamma_j+2\hbar)}\varkappa_j',$$

as required.

7.4.4. *Relation* (5.5). We first calculate the LHS:

$$\begin{aligned}
[b_i(u),b_i(v)] &= \sum_k \frac{\hbar(v-u)}{(u-\gamma_{i,k})(v-\gamma_{i,k})(u-\gamma_{i,k}+\hbar)(v-\gamma_{i,k}+\hbar)}\varkappa_{i,k}^2\\
&\quad+\sum_k \frac{\hbar(v-u)}{(u+\gamma_{i,k}+\hbar)(v+\gamma_{i,k}+\hbar)(u+\gamma_{i,k}+2\hbar)(v+\gamma_{i,k}+2\hbar)}(\varkappa_{i,k}')^2
\end{aligned}$$

$$+\sum_{k\neq l}\frac{(v-u)(\gamma_{i,k}-\gamma_{i,l})}{(u-\gamma_{i,k})(v-\gamma_{i,l})(v-\gamma_{i,k})(u-\gamma_{i,l})}\varkappa_{i,k}\varkappa_{i,l}$$
$$-\sum_{k\neq l}\varkappa'_{i,k}\varkappa'_{i,l}\frac{(v-u)(\gamma_{i,k}-\gamma_{i,l})}{(u+\gamma_{i,k})(v+\gamma_{i,l})(v+\gamma_{i,k})(u+\gamma_{i,l})}$$
$$+\sum_{k\neq l}\frac{(v-u)(\gamma_{i,k}+\gamma_{i,l}+\hbar)}{(u-\gamma_{i,k})(v-\gamma_{i,k})(u+\gamma_{i,l}+\hbar)(v+\gamma_{i,l}+\hbar)}[\varkappa_{i,k},\varkappa'_{i,l}]$$
$$+\sum_{k}\frac{2\gamma_{i,k}(v-u)}{(u^2-\gamma_{i,k}^2)(v^2-\gamma_{i,k}^2)}\varkappa_{i,k}\varkappa'_{i,k}$$
$$-\sum_{k}\frac{2(\gamma_{i,k}+\hbar)(v-u)}{(u^2-(\gamma_{i,k}+\hbar)^2)(v^2-(\gamma_{i,k}+\hbar)^2)}\varkappa'_{i,k}\varkappa_{i,k}.$$

Let us denote each of the seven summands above as $X_1,\cdots,X_7$, counting from the top.

We now pass to the RHS of (5.5):

$$b_i(u)-b_i(v)=\sum_k\frac{v-u}{(u-\gamma_{i,k})(v-\gamma_{i,k})}\varkappa_{i,k}+\frac{v-u}{(u+\gamma_{i,k}+\hbar)(v+\gamma_{i,k}+\hbar)}\varkappa'_{i,k},$$
$$(b_i(u)-b_i(v))^2=\sum_k\frac{(v-u)^2}{(u-\gamma_{i,k})(v-\gamma_{i,k})(u-\gamma_{i,k}+\hbar)(v-\gamma_{i,k}+\hbar)}\varkappa_{i,k}^2$$
$$+\sum_k\frac{(v-u)^2}{(u+\gamma_{i,k}+\hbar)(v+\gamma_{i,k}+\hbar)(u+\gamma_{i,k}+2\hbar)(v+\gamma_{i,k}+2\hbar)}(\varkappa'_{i,k})^2$$
$$+\sum_{k,l}\frac{(v-u)^2}{(u-\gamma_{i,k})(v-\gamma_{i,k})(u-\gamma_{i,l})(v-\gamma_{i,l})}\varkappa_{i,k}\varkappa_{i,l}$$
$$+\sum_{k,l}\varkappa'_{i,k}\varkappa'_{i,l}\frac{(v-u)^2}{(u+\gamma_{i,k})(v+\gamma_{i,k})(u+\gamma_{i,k})(v+\gamma_{i,k})}$$
$$+\sum_{k\neq l}\frac{(v-u)^2}{(u-\gamma_{i,k})(v-\gamma_{i,k})(u+\gamma_{i,l}+\hbar)(v+\gamma_{i,l}+\hbar)}[\varkappa_{i,k},\varkappa'_{i,l}]_+$$
$$+\sum_k\frac{(v-u)^2}{(u^2-\gamma_{i,k}^2)(v^2-\gamma_{i,k}^2)}\varkappa_{i,k}\varkappa'_{i,k}$$
$$+\sum_k\frac{(v-u)^2}{(u^2-(\gamma_{i,k}+\hbar)^2)(v^2-(\gamma_{i,k}+\hbar)^2)}\varkappa'_{i,k}\varkappa_{i,k}.$$

Let us denote each of the seven summands above as $Y_1,\cdots,Y_7$, counting from the top. We see immediately that $X_1=\frac{\hbar}{v-u}Y_1$ and $X_2=\frac{\hbar}{v-u}Y_2$. Relations (7.5)–(7.6) also imply that $X_3 = \frac{\hbar}{v-u} Y_3$, $X_4 = \frac{\hbar}{v-u} Y_4$, and $X_5 = \frac{\hbar}{v-u} Y_5$. Therefore, to establish (5.5), it suffices to show that

$$\sum_k \frac{2(\gamma_{i,k} - \frac{1}{2}\hbar)(v^2-u^2)}{(u^2-\gamma_{i,k}^2)(v^2-\gamma_{i,k}^2)} \varkappa_{i,k}\varkappa'_{i,k} - \sum_k \frac{2(\gamma_{i,k}+\frac{3}{2}\hbar)(v^2-u^2)}{(u^2-\xi_{i,k}^2)(v^2-\xi_{i,k}^2)} \varkappa'_{i,k}\varkappa_{i,k} = \\ = \hbar(\underline{z}_i(v) - \underline{z}_i(-u)). \tag{7.15}$$

Let us first calculate the LHS of (7.15). Using relations (7.7)–(7.8), we get

$$\text{LHS} = \sum_k \left(\frac{1}{u^2-\gamma_{i,k}^2} - \frac{1}{v^2-\gamma_{i,k}^2} \right) \frac{R_i(\gamma_{i,k}) \prod_{j=i\pm1} \prod_{l=1}^{m_j} (\gamma_{i,k}^2 - (\gamma_{j,l}+\frac{1}{2}\hbar)^2)}{2(\gamma_{i,k}+\frac{1}{2}\hbar) \prod_{l\neq k} (\gamma_{i,k}^2-\gamma_{i,l}^2)(\gamma_{i,k}^2-\xi_{i,l}^2)} \\ + \sum_k \left(\frac{1}{v^2-\xi_{i,k}^2} - \frac{1}{u^2-\xi_{i,k}^2} \right) \frac{R_i(\xi_{i,k}) \prod_{j=i\pm1} \prod_{l=1}^{m_j} (\xi_{i,k}^2 - (\gamma_{j,l}+\frac{1}{2}\hbar)^2)}{2(\gamma_{i,k}+\frac{1}{2}\hbar) \prod_{l\neq k} (\xi_{i,k}^2-\gamma_{i,l}^2)(\xi_{i,k}^2-\xi_{i,l}^2)}. \tag{7.16}$$

Next, we compute the RHS of (7.15). Applying partial fraction decomposition to the denominator of (7.14), we get

$$-\hbar z_i(u) = \sum_{k=1}^{m_i} \frac{R_i(u) \prod_{j=i\pm1} \prod_{l=1}^{m_j} (u^2 - (\gamma_{j,l}+\frac{1}{2}\hbar)^2)}{2(\gamma_{i,k}+\frac{1}{2}\hbar) \prod_{l\neq k}(\gamma_{i,k}^2-\gamma_{i,l}^2)(\gamma_{i,k}^2-\xi_{i,l}^2)} \frac{1}{u^2-\gamma_{i,k}^2} \\ - \sum_{k=1}^{m_i} \frac{R_i(u) \prod_{j=i\pm1} \prod_{l=1}^{m_j} (u^2 - (\gamma_{j,l}+\frac{1}{2}\hbar)^2)}{2(\gamma_{i,k}+\frac{1}{2}\hbar) \prod_{l\neq k}(\xi_{i,k}^2-\gamma_{i,l}^2)(\xi_{i,k}^2-\xi_{i,l}^2)} \frac{1}{u^2-\xi_{i,k}^2}. \tag{7.17}$$

The result now follows by comparing the 'u-part' of (7.16) with the principal part of (7.17), considered as Laurent series in u^{-2} (with analogous comparison for the 'v-part'). This follows from the fact that the principal parts of $\frac{p(\gamma_{i,k}^2)}{u^2-\gamma_{i,k}^2}$ and $\frac{p(u^2)}{u^2-\gamma_{i,k}^2}$ coincide for any polynomial $p(\cdot)$.

7.4.5. *Relation* (5.6). It suffices to prove (5.6) for $m_i = m_{i+1} = 1$, so we drop the second subscript on γ and $\varkappa$. The LHS is:

$$(u-v)[b_i(u), b_{i+1}(v)] = \frac{u-v}{(u-\gamma_i)(v-\gamma_{i+1})}[\varkappa_i, \varkappa_{i+1}] + [\varkappa'_i, \varkappa'_{i+1}] \frac{u-v}{(u+\gamma_i)(v+\gamma_{i+1})} \\ + \frac{u-v}{(u-\gamma_i)(v+\gamma_{i+1}+\hbar)}[\varkappa_i, \varkappa'_{i+1}] \\ + \frac{u-v}{(v-\gamma_{i+1})(u+\gamma_i+\hbar)}[\varkappa'_i, \varkappa_{i+1}].$$

On the other hand, on the RHS,

$$-\tfrac{1}{2}\hbar[b_i(u), b_{i+1}(v)]_+ = \frac{-\frac{1}{2}\hbar}{(u-\gamma_i)(v-\gamma_{i+1})}[\varkappa_i, \varkappa_{i+1}]_+ + [\varkappa'_i, \varkappa'_{i+1}]_+ \frac{-\frac{1}{2}\hbar}{(u+\gamma_i)(v+\gamma_{i+1})}$$

$$+\frac{-\frac{1}{2}\hbar}{(u-\gamma_i)(v+\gamma_{i+1}+\hbar)}[\varkappa_i,\varkappa'_{i+1}]_+$$
$$+\frac{-\frac{1}{2}\hbar}{(v-\gamma_{i+1})(u+\gamma_i+\hbar)}[\varkappa'_i,\varkappa_{i+1}]_+,$$

and

$$\begin{aligned}[b_i^{(0)},b_{i+1}(v)]+[b_{i+1}^{(0)},b_i(u)] &= \frac{u-v+\gamma_{i+1}-\gamma_i}{(u-\gamma_i)(v-\gamma_{i+1})}[\varkappa_i,\varkappa_{i+1}]+[\varkappa'_i,\varkappa'_{i+1}]\frac{u-v-\gamma_{i+1}+\gamma_i}{(u+\gamma_i)(v+\gamma_{i+1})}\\ &+\frac{u-v-\gamma_i-\gamma_{i+1}-\hbar}{(u-\gamma_i)(v+\gamma_{i+1}+\hbar)}[\varkappa_i,\varkappa'_{i+1}]\\ &+\frac{u-v+\gamma_i+\gamma_{i+1}+\hbar}{(v-\gamma_{i+1})(u+\gamma_i+\hbar)}[\varkappa'_i,\varkappa_{i+1}].\end{aligned}$$

The result is now implied by the identities

$$(\gamma_{i+1}-\gamma_i)[\varkappa_i,\varkappa_{i+1}]=\tfrac{1}{2}\hbar[\varkappa_i,\varkappa_{i+1}]_+,\qquad [\varkappa'_i,\varkappa'_{i+1}](\gamma_i-\gamma_{i+1})=\tfrac{1}{2}\hbar[\varkappa'_i,\varkappa'_{i+1}]_+,$$
$$(\gamma_{i+1}+\gamma_i+\hbar)[\varkappa_i,\varkappa'_{i+1}]=-\tfrac{1}{2}\hbar[\varkappa_i,\varkappa'_{i+1}]_+,\qquad (\gamma_{i+1}+\gamma_i+\hbar)[\varkappa'_i,\varkappa_{i+1}]=\tfrac{1}{2}\hbar[\varkappa_i,\varkappa'_{i+1}]_+,$$

which follow from (7.5)–(7.6).

7.4.6. *Relation* (5.7). Without loss of generality, we may assume $j=i+1$. Consider the LHS of (5.7) as a (degree 3) noncommutative polynomial in $\varkappa_{i,k},\varkappa'_{i,l},\varkappa_{i+1,m},\varkappa'_{i+1,n}$. First, we show that the sum of monomials *not* containing both $\varkappa_{i,k}$ and $\varkappa'_{i,k}$ (with the same second index) vanishes. Such monomials come in four different types:

(1) monomials containing $\varkappa_{i,k},\varkappa_{i,l},\varkappa_{i+1,m}$ or $\varkappa'_{i,k},\varkappa'_{i,l},\varkappa'_{i+1,m}$ (if $k=l$ then the associated variable occurs twice);

(2) monomials containing $\varkappa_{i,k},\varkappa_{i,l},\varkappa'_{i+1,m}$ or $\varkappa'_{i,k},\varkappa'_{i,l},\varkappa_{i+1,m}$ $(k\neq l)$;

(3) monomials containing $\varkappa_{i,k}$ with multiplicity 2 and $\varkappa'_{i+1,m}$, or $\varkappa'_{i,k}$ with multiplicity 2 and $\varkappa_{i+1,m}$;

(4) monomials containing $\varkappa_{i,k},\varkappa'_{i,l}$ $(k\neq l)$ and $\varkappa_{i+1,m}$ or $\varkappa'_{i+1,m}$.

The vanishing of the sum of all monomials of type 1 follows directly from the argument in the non-twisted case, i.e., [GKLO05, Lemma 3.1] or [BFN19, B(vi)]. For the other cases, we will need the following lemma.

Lemma 7.3. *The following identities hold:*

$$[\varkappa_{i,k},[\varkappa_{i,k},\varkappa'_{i+1,m}]]=0,$$
$$[\varkappa_{i,k},[\varkappa_{i,l},\varkappa'_{i+1,m}]]=\frac{-\hbar^2(\gamma_{i,k}+\gamma_{i,l}+2\gamma_{i+1,m}+2\hbar)}{(\gamma_{i,k}+\gamma_{j,m}+\frac{\hbar}{2})(\gamma_{i,l}+\gamma_{j,m}+\frac{\hbar}{2})(\gamma_{i,k}-\gamma_{i,l}+\hbar)}\varkappa_{i,k}\varkappa_{i,l}\varkappa'_{i+1,m},$$

$$[\varkappa_{i,k},[\varkappa'_{i,l},\varkappa_{i+1,m}]]=\frac{\hbar^2(\gamma_{i,l}-\gamma_{i,k}+2\gamma_{i+1,m}+\hbar)}{(\gamma_{i+1,m}-\gamma_{i,k}+\frac{\hbar}{2})(\gamma_{i,l}+\gamma_{i,k}+2\hbar)(\gamma_{i+1,m}+\gamma_{i,l}+\frac{3\hbar}{2})}\varkappa_{i,k}\varkappa'_{i,l}\varkappa_{i+1,m},$$

$$[\varkappa'_{i,k},[\varkappa_{i,l},\varkappa_{i+1,m}]]=\frac{-\hbar^2(\gamma_{i,k}-\gamma_{i,l}+2\gamma_{i+1,m}+\hbar)}{(\gamma_{i+1,m}-\gamma_{i,l}+\frac{\hbar}{2})(\gamma_{i,l}+\gamma_{i,k})(\gamma_{i+1,m}+\gamma_{i,k}+\frac{3\hbar}{2})}\varkappa'_{i,k}\varkappa_{i,l}\varkappa_{i+1,m},$$

for $k\neq l$.

Proof. The lemma follows by direct calculation using (7.9)–(7.11). □

Since

$$\text{(7.18)}\quad \operatorname{Sym}_{u_1,u_2}\left[\frac{1}{u_1-\gamma_{i,k}}\varkappa_{i,k},\left[\frac{1}{u_2-\gamma_{i,k}}\varkappa_{i,k},\frac{1}{t+\gamma_{i+1,m}+\hbar}\varkappa'_{i+1,m}\right]\right]=$$
$$=\operatorname{Sym}_{u_1,u_2}\frac{1}{(u_1-\gamma_{i,k}+\hbar)(u_2-\gamma_{i,k})(t+\gamma_{i,k}+\hbar)}\left[\varkappa_{i,k},\left[\varkappa_{i,k},\varkappa'_{i+1,m}\right]\right],$$

the first formula of Lemma 7.3 implies that the sum of all monomials of type 3, containing $\varkappa_{i,k}$ with multiplicity 2 and $\varkappa'_{i+1,m}$, on the LHS of (5.7), vanishes. It is clear that the other subcase, i.e., monomials of type 3 containing $\varkappa'_{i,k}$ with multiplicity 2 and $\varkappa_{i+1,m}$, can be handled using an analogous argument.

In the other cases, a similar calculation to (7.18) also shows that one can ignore the denominators, such as $\frac{1}{u_1-\gamma_{p,s}}$, in front of $\varkappa_{p,s},\varkappa'_{r,t}$. Let $k\neq l$. Then $\varkappa_{i,l}\varkappa_{i,k}\varkappa'_{i+1,m}=\frac{\gamma_{i,k}-\gamma_{i,l}-\hbar}{\gamma_{i,k}-\gamma_{i,l}+\hbar}\varkappa_{i,k}\varkappa_{i,l}\varkappa'_{i+1,m}$, and the second formula of Lemma 7.3 implies that $[\varkappa_{i,k},[\varkappa_{i,l},\varkappa'_{i+1,m}]]+[\varkappa_{i,l},[\varkappa_{i,k},\varkappa'_{i+1,m}]]=0$. Hence the sum of all monomials of type 2 on the LHS of (5.7) vanishes.

Again, let $k\neq l$. Then $\varkappa'_{i,l}\varkappa_{i,k}\varkappa_{i+1,m}=\frac{\gamma_{i,k}+\gamma_{i,l}}{\gamma_{i,k}+\gamma_{i,l}+2\hbar}\varkappa_{i,k}\varkappa'_{i,l}\varkappa_{i+1,m}$, and the third and fourth formulae of Lemma 7.3 imply that $[\varkappa_{i,k},[\varkappa'_{i,l},\varkappa_{i+1,m}]]+[\varkappa'_{i,l},[\varkappa_{i,k},\varkappa_{i+1,m}]]=0$. It follows that the sum of all monomials of type 4 on the LHS of (5.7) also vanishes.

We will now prove that the sum of the remaining monomials equals the RHS of (5.7). We will need the following lemma.

Lemma 7.4. *The following identities hold:*

$$\varkappa_{i,k}\varkappa'_{i,k}\varkappa_{i+1,l}-2\varkappa_{i,k}\varkappa_{i+1,l}\varkappa'_{i,k}+\varkappa_{i+1,l}\varkappa_{i,k}\varkappa'_{i,k}=\frac{\frac{1}{2}\hbar S^{i+1,l}_{i,k}}{\gamma_{i,k}+\frac{1}{2}\hbar}\varkappa_{i+1,l},$$

$$\varkappa'_{i,k}\varkappa_{i,k}\varkappa_{i+1,l}-2\varkappa'_{i,k}\varkappa_{i+1,l}\varkappa_{i,k}+\varkappa_{i+1,l}\varkappa'_{i,k}\varkappa_{i,k}=\frac{\frac{1}{2}\hbar T^{i+1,l}_{i,k}}{\gamma_{i,k}+\frac{1}{2}\hbar}\varkappa_{i+1,l},$$

$$\varkappa_{i,k}\varkappa'_{i,k}\varkappa'_{i+1,l}-2\varkappa_{i,k}\varkappa'_{i+1,l}\varkappa'_{i,k}+\varkappa'_{i+1,l}\varkappa_{i,k}\varkappa'_{i,k}=\frac{\frac{1}{2}\hbar S^{i+1,l}_{i,k}}{\gamma_{i,k}+\frac{1}{2}\hbar}\varkappa'_{i+1,l},$$

$$\varkappa'_{i,k}\varkappa_{i,k}\varkappa'_{i+1,l}-2\varkappa'_{i,k}\varkappa'_{i+1,l}\varkappa_{i,k}+\varkappa'_{i+1,l}\varkappa'_{i,k}\varkappa_{i,k}=\frac{\frac{1}{2}\hbar T^{i+1,l}_{i,k}}{\gamma_{i,k}+\frac{1}{2}\hbar}\varkappa'_{i+1,l}.$$

Proof. We have

$$\varkappa_{i,k}\varkappa'_{i,k}\varkappa_{i+1,l} = \frac{\gamma_{i,k}^2 - (\gamma_{i+1,l} + \frac{1}{2}\hbar)^2}{4(\gamma_{i,k} - \frac{1}{2}\hbar)(\gamma_{i,k} + \frac{1}{2}\hbar)} S_{i,k}^{i+1,l}\varkappa_{i+1,l},$$

$$\varkappa_{i,k}\varkappa_{i+1,l}\varkappa'_{i,k} = \frac{(\gamma_{i,k} - \frac{1}{2}\hbar)^2 - \gamma_{i+1,l}^2}{4(\gamma_{i,k} - \frac{1}{2}\hbar)(\gamma_{i,k} + \frac{1}{2}\hbar)} S_{i,k}^{i+1,l}\varkappa_{i+1,l},$$

$$\varkappa_{i+1,l}\varkappa_{i,k}\varkappa'_{i,k} = \frac{\gamma_{i,k}^2 - (\gamma_{i+1,l} - \frac{1}{2}\hbar)^2}{4(\gamma_{i,k} - \frac{1}{2}\hbar)(\gamma_{i,k} + \frac{1}{2}\hbar)} S_{i,k}^{i+1,l}\varkappa_{i+1,l}.$$

The first identity of the lemma now follows from the fact that

$$\big(\gamma_{i,k}^2 - (\gamma_{i+1,l} + \tfrac{1}{2}\hbar)^2\big) - 2\big((\gamma_{i,k} - \tfrac{1}{2}\hbar)^2 - \gamma_{i+1,l}^2\big) + \big(\gamma_{i,k}^2 - (\gamma_{i+1,l} - \tfrac{1}{2}\hbar)^2\big) = 2\hbar(\gamma_{i,k} - \tfrac{1}{2}\hbar).$$

An analogous argument establishes the other identities. □

Lemma 7.4 implies that the LHS of (5.7) is equal to

$$\mathrm{LHS} = \sum_{u=u_1,u_2}\left(\sum_{k,l}\frac{\hbar u S_{i,k}^{i+1,l}}{(u^2 - \gamma_{i,k}^2)(\gamma_{i,k} + \frac{1}{2}\hbar)} b_{i+1}^l(t) + \sum_{k,l}\frac{\hbar u T_{i,k}^{i+1,l}}{(u^2 - \xi_{i,k}^2)(\gamma_{i,k} + \frac{1}{2}\hbar)} b_{i+1}^l(t)\right),$$

where

$$b_{i+1}^l(t) = \frac{1}{t - \gamma_{i+1,l}}\varkappa_{i+1,l} + \frac{1}{t + \gamma_{i+1,l} + \hbar}\varkappa'_{i+1,l}.$$

On the other hand, using (7.14), and the fact that

$$((t-\hbar)\, z_i(u)\varkappa_{i+1,l} - (t+\hbar)\,\varkappa_{i+1,l} z_i(u)) =$$
$$= \frac{\hbar\big((-2u^2 + \frac{1}{2}\hbar^2) + 2\gamma_{i+1,l}(\gamma_{i+1,l} - t)\big)}{u^2 - (\gamma_{i+1,l} + \frac{1}{2}\hbar)^2} z_i(u)\varkappa_{i+1,l},$$
$$\big((t-\hbar)\, z_i(u)\varkappa'_{i+1,l} - (t+\hbar)\,\varkappa'_{i+1,l} z_i(u)\big) =$$
$$= \frac{\hbar\big((-2u^2 + \frac{1}{2}\hbar^2) + 2\xi_{i+1,l}(\xi_{i+1,l} + t)\big)}{u^2 - (\gamma_{i+1,l} + \frac{1}{2}\hbar)^2} z_i(u)\varkappa'_{i+1,l},$$

we conclude that

$$\frac{(t-\hbar)\, z_i(u) b_{i+1}(t) - (t+\hbar)\, b_{i+1}(t) z_i(u)}{4u^2 - \hbar^2} =_t$$
$$=_t -\sum_l R_i(u)\frac{\prod_{p=1}^{m_{i-1}}(u^2 - (\gamma_{i-1,p} + \frac{1}{2}\hbar)^2)\prod_{l\neq p=1}^{m_{i+1}}(u^2 - (\gamma_{i+1,k} + \frac{1}{2}\hbar)^2)}{2\prod_{k=1}^{m_i}(u^2 - \gamma_{i,k}^2)(u^2 - \xi_{i,k}^2)} b_{i+1}^l(t),$$

where $=_t$ denotes the equality of t-principal parts. A partial fraction decomposition argument, analogous to §7.4.4, completes the proof of (5.7). This also concludes the proof of Theorem 7.2.

7.5. **Symmetrized representation.** Our definition of the operators (7.2)–(7.3) is obviously asymmetric. This asymmetry can be resolved at the cost of working in a certain quadratic extension of $D^\lambda_{\mu,\hbar}$. More precisely, let $\breve{D}^\lambda_{\mu,\hbar}$ be the $\mathbb{C}[\hbar]$-algebra generated by

- $\gamma_{i,k}, \beta^{\pm 1}_{i,k}$ (for $i \in \mathbb{I}$ and $1 \leq k \leq m_i$),
- $((\gamma_{i,k} + r\hbar)^2 - (\gamma_{i,l} + s\hbar)^2)^{-1}$, (for $k \neq l$ and $r, s \in \mathbb{Z}$),
- $(\gamma_{i,k} \pm \frac{r}{2}\hbar)^{-1}$ (for $r \in \mathbb{Z}$),
- $(\gamma_{i,k} - \gamma_{i\pm 1,l} + \frac{r}{2}\hbar)^{\frac{1}{2}}$ and $(\gamma_{i,k} + \gamma_{i\pm 1,l} + \frac{r}{2}\hbar)^{\frac{1}{2}}$, (for $r \in \mathbb{Z}$),

subject to the relations

$$[\beta^{\pm 1}_{i,k}, \gamma_{j,l}] = \pm \delta_{ij}\delta_{kl}\hbar\beta^{\pm 1}_{i,k}, \qquad [\gamma_{i,k}, \gamma_{j,l}] = 0 = [\beta_{i,k}, \beta_{j,l}], \qquad \beta^{\pm 1}_{i,k}\beta^{\mp 1}_{i,k} = 1.$$

Let us fix a root $\sqrt{-1}$ of -1. In the definition above, we choose the roots consistently, so that, e.g.,

$$(\gamma_{i,k} - \gamma_{i+1,l})^{\frac{1}{2}} = \sqrt{-1}(\gamma_{i+1,l} - \gamma_{i,k})^{\frac{1}{2}}.$$

Then we can define

$$\breve{\varkappa}_{i,k} = (R_i(\gamma_{i,k}))^{\frac{1}{2}} \frac{\prod_{j\in\{i\pm 1\}}\prod_{l=1}^{m_{i-1}}(\gamma_{i,k}^2 - (\gamma_{j,l} + \frac{1}{2}\hbar)^2)^{\frac{1}{2}}}{2(\gamma_{i,k} - \frac{1}{2}\hbar)\prod_{l\neq k}(\gamma_{i,k}^2 - \gamma_{i,l}^2)}\beta^{-1}_{i,k} \in \breve{D}^\lambda_{\mu,\hbar},$$

$$\breve{\varkappa}'_{i,k} = (R_i(\xi_{i,k}))^{\frac{1}{2}} \frac{\prod_{j\in\{i\pm 1\}}\prod_{l=1}^{m_{i-1}}(\xi_{i,k}^2 - (\gamma_{j,l} + \frac{1}{2}\hbar)^2)^{\frac{1}{2}}}{2(\gamma_{i,k} + \frac{3}{2}\hbar)\prod_{l\neq k}(\xi_{i,k}^2 - \xi_{i,l}^2)}\beta_{i,k} \in \breve{D}^\lambda_{\mu,\hbar}.$$

One easily checks that Lemma 7.1 still holds if we replace $\varkappa_{i,k} \leftrightarrow \breve{\varkappa}_{i,k}$ and $\varkappa'_{i,k} \leftrightarrow \breve{\varkappa}'_{i,k}$. In particular, carrying out appropriate modifications throughout §7.4, we get the following version of Theorem 7.2.

Corollary 7.5. *The assignment*

$$b_i(u) \mapsto \sum_{k=1}^{m_i} \frac{1}{u - \gamma_{i,k}}\breve{\varkappa}_{i,k} + \frac{1}{u + \gamma_{i,k} + \hbar}\breve{\varkappa}'_{i,k},$$

$$h_i(u) \mapsto u^{-4m_1}\prod_{j=1}^{i-1} R_j(u - \tfrac{i-1-j}{2}\hbar)\frac{\prod_{k=1}^{m_i}(u^2 - (\gamma_{i,k} + \frac{1}{2}\hbar)^2)}{\prod_{l=1}^{m_{i-1}}(u - \gamma_{i-1,l})(u + \gamma_{i-1,l} + \hbar)}$$

defines a homomorphism $\breve{\Phi}^\lambda_\mu \colon {}^{\mathsf{tw}}\widetilde{Y}_{\mu,\hbar} \to \breve{D}^\lambda_{\mu,\hbar}$.

Remark 7.6. The symmetric formulation with square roots is somewhat more natural from the point of view of Gelfand–Tsetlin theory, see, e.g., [GK91] and [LP25, Theorems 4.3, 6.1].

7.6. **ABCD formulation.** Throughout this subsection, let $\mu = 0$. With a view to geometric applications, it is convenient to reformulate the twisted GKLO representations in terms of the ABCD presentation from §6.

Corollary 7.7. *The twisted GKLO homomorphism* $\Phi_0^\lambda \colon {}^{\mathsf{tw}}\widetilde{Y}_{0,\hbar} \to D_{0,\hbar}^\lambda$ *is uniquely determined by the following formulae:*

$$\widetilde{A}_i(u) \mapsto u^{-2m_i} \prod_{k=1}^{m_i} (u^2 - (\gamma_{i,k} + \tfrac{1}{2}\hbar)^2),$$

$$\widetilde{B}_i(u) \mapsto -u^{-2m_i} \sum_{k=1}^{m_i} \prod_{k \neq l=1}^{m_i} (u^2 - (\gamma_{i,l} + \tfrac{1}{2}\hbar)^2) \Big((u - \gamma_{i,k} - \tfrac{1}{2}\hbar)\varkappa_{i,k} + (u + \gamma_{i,k} + \tfrac{1}{2}\hbar)\varkappa'_{i,k} \Big),$$

$$\widetilde{C}_i(u) \mapsto u^{-2m_i} \sum_{k=1}^{m_i} \Big(\varkappa_{i,k}(u - \gamma_{i,k} - \tfrac{1}{2}\hbar) + \varkappa'_{i,k}(u + \gamma_{i,k} + \tfrac{1}{2}\hbar) \Big) \prod_{k \neq l=1}^{m_i} (u^2 - (\gamma_{i,l} + \tfrac{1}{2}\hbar)^2).$$

The symmetrized twisted GKLO homomorphism $\breve{\Phi}_0^\lambda \colon {}^{\mathsf{tw}}\widetilde{Y}_{0,\hbar} \to \breve{D}_{0,\hbar}^\lambda$ *is given by the same formulae, with replacements* $\varkappa_{i,k} \leftrightarrow \breve{\varkappa}_{i,k}$ *and* $\varkappa'_{i,k} \leftrightarrow \breve{\varkappa}'_{i,k}$.

Proof. This follows directly from Theorem 7.2, Corollary 7.5, as well as comparing formulae (6.1) and (7.14) (the former needs to be twisted by the action of the central element $c(u) \mapsto u^{-2\lambda_i} R_i(u)$ from (5.8)). The normalizing factor u^{-2m_i} is used to ensure $\widetilde{A}_i(u)$ is a series of the form $\widetilde{A}_i(u) = \sum_{k \geq 0} A_i^{(k)} u^{-k}$. □

We can now, at least partially, characterize the kernel of the twisted GKLO homomorphism.

Corollary 7.8. *The following elements are in the kernel of* Φ_0^λ*:*

- $\widetilde{A}_i^{(r)}, \widetilde{B}_i^{(r)}, \widetilde{C}_i^{(r)}$ *for* $r > 2m_i$.

Proof. The statement is immediate from Corollary 7.7. □

8. Geometric realisations

Here we show that the shifted twisted Yangians from Section 5 and their truncations defined in Section 7 quantise the Poisson structures discussed in Section 3.

8.1. **Quantisations via RTT generators.** It follows from the discussion in Section 5.4 that ${}^{\mathsf{tw}}\mathbf{Y}_0 / \hbar\, {}^{\mathsf{tw}}\mathbf{Y}_0$ is generated as a $\mathbb{C}$-algebra by elements $s_{ij}^{(r)}$ with $1 \leq i, j \leq n$ and $r \geq 1$ subject to relations $s_{ij}^{(r)} = s_{ji}^{(r)}(-1)^r$. If $s_{ij}(z) := \sum_{r \geq 0} s_{ij}^{(r)} z^{-r}$ for

$$s_{ij}^{(0)} := \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

then the Poisson bracket on ${}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0$ induced as in Section 2.3 is described by

$$(u^2-v^2)\{s_{ij}(u),s_{kl}(v)\} = (u+v)\left(s_{kj}(u)s_{il}(v)-s_{il}(u)s_{kj}(v)\right) - (u-v)\left(s_{ik}(u)s_{jl}(v)-s_{lj}(u)s_{ki}(v)\right) \tag{8.1}$$

for variables u, v.

Proposition 8.1. *The map of $\mathbb{C}$-algebras*

$${}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0 \to \mathcal{O}(K_0\backslash \operatorname{Gr}_0)$$

given by $s_{ij}^{(r)} \mapsto \Delta_{ji}^{\tau,(r)}$ (recall Notation 3.10) is a Poisson isomorphism.

Proof. Clearly the given map is an isomorphism of commutative rings. Recall from Section 3 that $\{-,-\}$ denotes the Poisson bracket on $\mathcal{G}_0$, while $\{-,-\}_\tau$ denotes the Poisson bracket on $\mathcal{O}(K_0\backslash\mathcal{G}_0)$ obtained via Ψ from the Dirac reduction of $\{-,-\}$ on $\mathcal{G}_0^{\tau=1}$. Lemma 3.1 and (3.3) then combine to give

$$\begin{aligned}(u^2-v^2)\{\Delta_{ij}^\tau(u),\Delta_{kl}^\tau(v)\}_\tau =&(u^2-v^2)\{\Delta_{ij}(u),\Delta_{kl}(v)\}\circ\Psi\\ &+(u^2-v^2)\{\Delta_{ji}(-u),\Delta_{kl}(v)\}\circ\Psi\\ =&(u+v)\left(\Delta_{il}^\tau(u)\Delta_{kj}^\tau(v)-\Delta_{kj}^\tau(u)\Delta_{il}^\tau(v)\right)\\ &-(u-v)\left(\Delta_{jl}^\tau(-u)\Delta_{ki}^\tau(v)-\Delta_{ki}^\tau(-u)\Delta_{jl}^\tau(v)\right)\\ =&(u+v)\left(\Delta_{il}^\tau(u)\Delta_{kj}^\tau(v)-\Delta_{kj}^\tau(u)\Delta_{il}^\tau(v)\right)\\ &-(u-v)\left(\Delta_{lj}^\tau(u)\Delta_{ki}^\tau(v)-\Delta_{ik}^\tau(u)\Delta_{jl}^\tau(v)\right)\end{aligned}$$

Comparing with (8.1) shows that $s_{ij}^{(r)} \mapsto \Delta_{ij}^{(r)}$ as anti Poisson (i.e. maps the bracket on ${}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0$ onto -1 times the bracket on $\mathcal{O}(K_0\backslash\mathcal{G}_0)$). Since the association $s_{ij}^{(r)} \mapsto s_{ji}^{(r)}$ is an anti-automorphism of ${}^{\mathsf{tw}}\mathbf{Y}_0$, see [Mol07, Proposition 2.3.4], the claim follows. □

Remark 8.2. The existence of the isomorphism in Proposition 8.3 can also understood conceptually from the viewpoint of [KWWY14]. Specifically, [KWWY14, Theorem 3.9] produces a Poisson isomorphism $\phi : \mathcal{O}(\operatorname{Gr}_0) \to \mathbf{Y}_0/\hbar\mathbf{Y}_0$ where $\mathbf{Y}_0$ denotes the $\mathbb{C}[\hbar]$ form of the $\mathfrak{sl}_n$ Yangian. This isomorphism identifies the universal matrix on $\operatorname{Gr}_0 \cong \mathcal{G}_0$ with the transpose of the matrix $T(z) \in \operatorname{Mat}(\mathbf{Y}_0/\hbar\mathbf{Y}_0)[[z^{-1}]]$ of RTT generators. As a consequence, the isomorphism from Proposition 8.1 fits into the following commutative diagram

$$\begin{array}{ccc}\mathcal{O}(K_0\backslash \operatorname{Gr}_0) & \longrightarrow & \mathcal{O}(\operatorname{Gr}_0)\\ \downarrow & & \downarrow\\ {}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0 & \longrightarrow & \mathbf{Y}_0/\hbar\mathbf{Y}_0\end{array}$$

in which the lower horizontal embedding identifies ${}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0$ with the subalgebra of $\mathbf{Y}_0/\hbar\mathbf{Y}_0$ generated over $\mathbb{C}$ by the coefficients of the entries inside $T(z)T^t(-z)$ [Mol07, Theorem 2.4.3].

8.2. Quantisations in the shifted setting.

Proposition 8.3. *Let $\mu \in X_*(T)$ be dominant and recall the shift homomorphism ${}^{\mathsf{tw}}\iota(\mu) : {}^{\mathsf{tw}}\mathbf{Y}_\mu \to {}^{\mathsf{tw}}\mathbf{Y}_0$. Then there is a commutative diagram of graded Poisson $\mathbb{C}$-algebras*

$$\begin{array}{ccc} \mathcal{O}(K_0\backslash\mathcal{G}_0/\mathcal{U}^{-,\mu}) & \longrightarrow & \mathcal{O}(K_0\backslash\mathcal{G}_0) \\ \downarrow\wr & & \downarrow\wr \\ {}^{\mathsf{tw}}\mathbf{Y}_\mu/\hbar{}^{\mathsf{tw}}\mathbf{Y}_\mu & \xrightarrow{{}^{\mathsf{tw}}\iota(\mu)} & {}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0 \end{array}$$

whose right vertical arrow is the isomorphism in Proposition 8.3.

Proof. The image of ${}^{\mathsf{tw}}\mathbf{Y}_\mu/\hbar{}^{\mathsf{tw}}\mathbf{Y}_\mu$ inside ${}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0$ under the shift homomorphism is Poisson generated by the images modulo $\hbar$ of the element $\hbar b_i^{(r)} \in {}^{\mathsf{tw}}\mathbf{Y}_0$ for $r > \langle \alpha_i^\vee, \mu\rangle$ and the $\hbar z_i^{(r)} \in {}^{\mathsf{tw}}\mathbf{Y}_0$ for $r > 0$. On the other hand, Proposition 6.1 implies that the isomorphism in Proposition 8.3 identifies the series

$$\hbar z_i(u) = \frac{A_{i-1}(u)A_{i+1}(u)}{A_i(u)^2}, \qquad \hbar b_i(u) = B_i(u)A_i(u)$$

for $A_i(u) = \sum_{r\geq 0} A_i^{(r)}u^{-r}, B_i(u) = \sum_{r>0} B_i^{(r)}u^{-r}$ and $A_i^{(r)}, B_i^{(r)}$ as defined in Section 3.4. It is easy to see that each coefficient in $A_i(u)$ lies inside the subring $\mathcal{O}(K_0\backslash\mathcal{G}_0/\mathcal{U}^{-,\mu})$ since the trailing principal minor of any $g \in G$ is invariant under left multiplication by U^+ and right multiplication by U^-. On the other hand, if $K_0x \in K_0\backslash\mathcal{G}_0$ then $b_i(K_0x)$ is the $i, i+1$-th entry of the matrix $f \in \mathcal{U}_0^-$ obtained by factoring $\tau(x)x = edf$ as in (2.1). It is easy to see that the coefficients in this series of degree $< \langle\alpha_i^\vee, \mu\rangle$ are invariant under the right action of $\mathcal{U}^{-,\mu}$. We conclude that, under the identification of Proposition 8.3, the image of ${}^{\mathsf{tw}}\mathbf{Y}_\mu/\hbar{}^{\mathsf{tw}}\mathbf{Y}_\mu$ lies inside $\mathcal{O}(K_0\backslash\mathcal{G}_0/\mathcal{U}^{-,\mu})$.

It remains to show this inclusion is an equality. For this it suffices to observe that both have Hilbert series for the loop grading given by

$$\left(\prod_{\alpha^\vee\in\Delta^+}\prod_{q>\langle\alpha^\vee,\mu\rangle}^{\infty}\frac{1}{(1-t^q)}\right)\left(\prod_{j=1}^{\infty}\frac{1}{(1-t^{2j})^{n-1}}\right).$$

For $\mathcal{O}(K_0\backslash\mathcal{G}_0/\mathcal{U}^{-,\mu})$ this is easily seen using the isomorphism

$$K_0\backslash\mathcal{G}_0/\mathcal{U}^{-,\mu} \cong \left(\mathcal{U}^+_{-\mu}\times\mathcal{T}_0\times\mathcal{U}^-_\mu\right)^{\tau=1}$$

in (3.2), while for ${}^{\mathsf{tw}}\mathbf{Y}_\mu/\hbar{}^{\mathsf{tw}}\mathbf{Y}_\mu$, it follows from the description of the PBW basis in Proposition 5.3. □

8.3. **Quantisations of truncations.** Here we take $\mu = 0$ and consider $\lambda \in X_*(T)^+$. Then Theorem 7.2 produces a surjection ${}^{\mathsf{tw}}\mathbf{Y}_0 \to {}^{\mathsf{tw}}\mathbf{Y}_0^\lambda$, and hence a surjection ${}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0 \to {}^{\mathsf{tw}}\mathbf{Y}_0^\lambda/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0^\lambda$.

Theorem 8.4. *There is a commutative diagram*

$$\begin{array}{ccc} {}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0 & \xrightarrow{\text{Prop 8.3}} & \mathcal{O}(K_0\backslash\mathcal{G}_0) \\ \downarrow & & \downarrow \\ {}^{\mathsf{tw}}\mathbf{Y}_0^\lambda/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0^\lambda & \longrightarrow & \mathcal{O}(\mathcal{S}_0^{\leq -w_0(2\lambda)}) \end{array}$$

whose bottom horizontal arrow is an isomorphism modulo the ideal of nilpotent elements in ${}^{\mathsf{tw}}\mathbf{Y}_0^\lambda/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0^\lambda$.

Proof. Write I for the Poisson ideal obtained as the kernel of the surjection $\mathcal{O}(K_0\backslash\mathcal{G}_0) \cong {}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0 \to {}^{\mathsf{tw}}\mathbf{Y}_0^\lambda/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0^\lambda$. From Corollary 7.8 we see that

$$\Psi_0^\lambda(A_i^{(r)}) = 0$$

for $r > m_i := \langle \omega_i, 2\lambda\rangle$. Since $m_i = \langle \omega_i, -w_0(-w_0(2\lambda))\rangle$ it follows that I contains the ideal described in Corollary 3.13 when applied to the dominant coweight $-w_0(2\lambda)$. Thus, the vanishing locus $V(I) \subset K_0\backslash\mathcal{G}_0$ of I is contained in $\mathcal{S}_0^{\leq -w_0(2\lambda)}$. If this containment were strict then I could not contain any $A_i^{(m_i)}$. Indeed, Proposition 3.11 asserts that these functions are units in $\mathcal{O}(\mathcal{S}_0^{\leq -w_0(2\lambda)})$. However, it is clear from Corollary 7.7 that $\Psi_0^\lambda(A_i^{(m_i)}) \neq 0$, so we are done. □

Theorem 8.5. *Suppose that Conjecture 3.14. Then:*

1. *The bottom horizontal arrow in Theorem 8.4 is an isomorphism.*
2. ${}^{\mathsf{tw}}\mathbf{Y}_0^\lambda$ *is the quotient of* ${}^{\mathsf{tw}}\mathbf{Y}_0$ *by the two sided ideal generated by the* $\hbar A_i^{(r)}$*'s for* $r > r_i = \langle \omega_i, 2\lambda\rangle$ *and the* $\hbar B_i^{(r_i+1)}$ *for each* $1 \leq i \leq n-1$.

Proof. The argument is identical to that proving [KWWY14, Theorem 4.10]. Let $K \subset {}^{\mathsf{tw}}\mathbf{Y}_0$ denote the two sided ideal described in the theorem. Then

$$K/\hbar K \subset I \subset J_0^\lambda$$

for J_0^λ the ideal defining $\mathcal{S}_0^{\leq -w_0(2\lambda)}$ and I the kernel of ${}^{\mathsf{tw}}\mathbf{Y}_0/\hbar{}^{\mathsf{tw}}\mathbf{Y}_0 \to \mathcal{O}(\mathcal{S}_0^{\leq -w_0(2\lambda)})$. On the other hand, the proof of Theorem 8.4 goes through with I replaced by $K/\hbar K$ since Corollary 7.8 clearly shows that

$$\Psi_0^\lambda(B_i^{(r)}) = 0$$

for $r > \langle \omega_i, 2\lambda\rangle$. Thus, $K/\hbar K$ has radical equal to J_0^λ. But Conjecture 3.14 asserts that $K/\hbar K$ is already reduced, and hence $K/\hbar K = I = J_0^\lambda$ which gives part (1). Part (2) then follows from an application of Nakayama's lemma. □

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